On Venn Diagrams and the Counting of Regions

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Most teachers utilize Venn diagrams to teach the basics of set operations, using circles (with their interiors and exteriors) as subsets of the universal set (for traditional reasons sometimes depicted as the inside of a rectangle). For this purpose diagrams using two or three circles usually suffice. Hence the article "Is the Venn diagram good enough?" by Mou-Ling Kung and George C. Harrison [CMJ 15 (January 1984) 48–50] is instructive: Professor I. M. Tweedy is correct that $n^2 - n + 2$ *is the maximum number of disjoint regions in the plane that can be formed by n circles*. But there is more here than meets the eye. To begin with let us recall [CMJ 15 (June 1984) 238–242] the standard definition of a Venn diagram.

A family $\mathscr{T} = \{A_1, A_2, \ldots, A_n\}$ of n simple closed curves A_j in the plane is called *independent* if every intersection of the type

$$X_1 \cap X_2 \cap \cdots \cap X_n \qquad (*)$$

is a nonempty set, where each X_j is chosen to be either the *interior* A_j^i or the *exterior* A_j^e of the curve A_j. An independent family $\mathscr{T}$ is called a *Venn diagram* if each set of the form (*) is connected—that is, not the union of two disjoint, relatively open sets. The independent family $\mathscr{T} = \{A_1, A_2\}$ in Figure 1 is not a Venn diagram since the set $A_1^i \cap a_2^i$ is not connected; the family $\mathscr{T} = \{A_1, A_2, A_3\}$ in Figure 2 is not independent since $A_1^e \cap A_2^i \cap A_3^e = \emptyset$. The five equilateral triangles in Figure 3 form a rather symmetric Venn diagram, as can easily be checked.

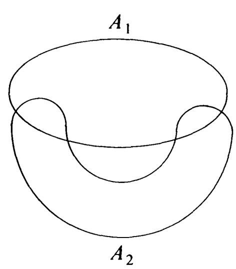

Figure 1.

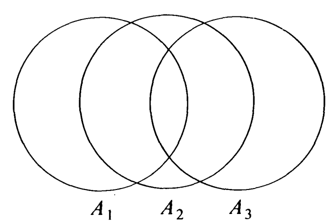

Figure 2.

Let us now return to the classroom where Professor Tweedy is "showing" the construction of a Venn diagram with four circles. Before being saved by the bell, he is counting the regions formed by various sets of four circles, ending each time with 14 (instead of the hoped-for 16). The impression is left that his problem would have been solved had he only been able to come up with an example in which the number is correct. However, since with three circles we can construct Venn diagrams, it is of interest to note that we can also find sets of three circles which define the correct number (eight) of regions, such that the circles *do not* form a Venn diagram (nor an independent set, see Figure 2) since two of the regions *together* form one of the sets (*). Thus, the finding of the right *number* of regions is a necessary condition for Venn diagrams, but not sufficient.

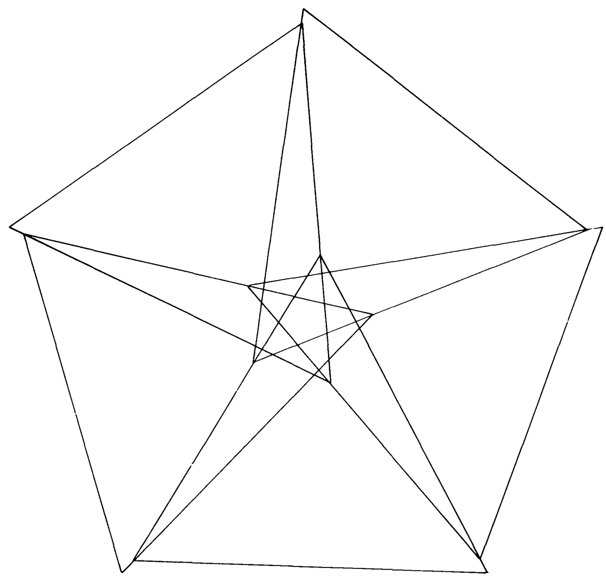

Figure 3. A Venn diagram formed by five congruent equilateral triangles. Probably no family of six triangles forms a Venn diagram, but this has not been proved so far.

Now let us consider, instead of families of circles, families of simple closed curves which pairwise intersect in at most two points. Here, again, hides something quite unobvious. For each family of few (say 3 or 4) such curves, one can find a family of circles which intersect in the same way the curves do (in other words, the cell complexes they define in the plane are isomorphic). However, with sufficiently many (say 6) curves it is possible to construct some intersection patterns which are *not* isomorphic to *any* intersection pattern of a family of circles (this follows from various "closure" theorems for circles, see Figure 4)—thus there is genuine widening of scope by substituting curves for circles in the estimate of the number of regions.

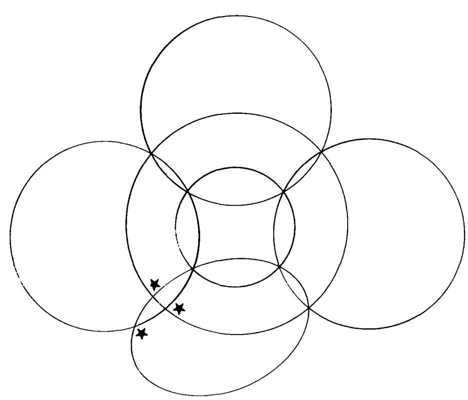

Figure 4. A family of six simple closed curves, pairwise intersecting in at most two points, which has an intersection pattern that cannot be obtained by six circles.*

Professor Tweedy's result concerning circles can be extended to more general families of curves:

$n^2 - n + 2$ is the maximum number of disjoint regions that can be formed in the plane by n simple closed curves which pairwise meet in at most two points.

This can be shown very easily by an inductive argument (quite similar to Professor Tweedy's count in his particular example): If $f(n)$ denotes the maximal possible number of regions determined by any such family of n curves, then adjoining an additional curve can add at most $2n$ to the number of regions, hence $f(n+1) \leqslant 2n + f(n)$ for all n; since $f(1) = 2$ we get $f(n) \leqslant n^2 - n + 2$. Professor Tweedy's *example* can now be used to complete the proof by showing that equality is attainable for each n.

It follows, in particular, that it is not possible to form a Venn diagram with any four simple curves which intersect pairwise in at most two points.

With ellipses (even with congruent ellipses) it is possible to form Venn diagrams for four sets, as well as for five sets—but not for six or more. Venn diagrams for arbitrarily many sets can be constructed in a variety of ways. For these and related results see "Venn Diagrams and Independent Families of Sets" [*Math. Mag.* 48 (January 1975) 12–22], "A Venn Diagram of Five Triangles" [*Math. Mag.* 55 (November 1982) 311], and "The Construction of Venn Diagrams" [CMJ 15 (June 1984) 238–247].

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*This follows from the classical theorem of Miquel (1844): if four circles are arranged in sequence, each two successive circles intersecting in pairs of points, and if a circle passes through one point of each of the four pairs of intersection points, then the remaining four intersection points lie on another circle (or on a straight line). For this and related results see, for example, J. L. Coolidge, "A Treatise on the Circle and the Sphere," Clarendon Press, Oxford 1916, p. 86, or R. A. Johnson, Advanced Euclidean Geometry, Dover, New York 1960, p. 135. If it were possible to replace the curves by circles with the same intersection pattern, the three intersection points (marked by asterisks in Figure 4) of pairs of circles would have to coincide (by Miquel's theorem) at a triple intersection point.