But (5) is true by virtue of (3) and the identity

$$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B.$$

The polar coordinate proof of the hyperbola's reflection property is similar and therefore left to the reader. (Note that the branch of the hyperbola not encompassing F' is generated by negative values of r'.)

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A Sequel to "Another Way of Looking at n!"

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In the *TYCMJ*'s November 1980 Classroom Capsules Column, Davis Hsu observed that $1/n!$ is the content (volume) of the n-dimensional simplex determined by

$$x_1 + x_2 + \cdots + x_n \leqslant 1, \qquad x_i \geqslant 0\ (1 \leqslant i \leqslant n).$$

Going a step further, we show that the content of the polytope P_n determined by

$$|x_1| + |x_2| + \cdots + |x_n| + |x_1 + x_2 + \cdots + x_n| \leqslant 2$$

is $\binom{2n}{n}/n!$. Figures 1 and 2 depict P_2 and P_3, respectively.

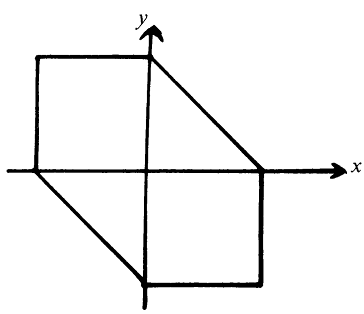

Figure 1.

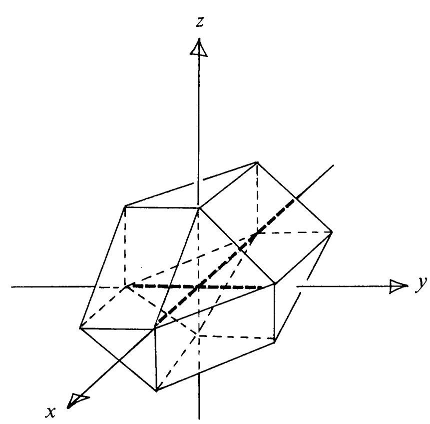

Figure 2.

In 2-space (the plane) there are 4 quadrants. In 3-space there are 8 octants. In n-space there are 2^n "octants." Indeed, for any k $(0 \leqslant k \leqslant n)$ there are $\binom{n}{k}$ octants . . . one for each choice of the k coordinates that are nonnegative (the other $n - k$ coordinates are nonpositive). For example, one such octant with k nonnegative and $n - k$ nonpositive coordinates is determined by

$$x_i \geqslant 0 \quad (1 \leqslant i \leqslant k) \quad \text{and} \quad x_i \leqslant 0 \quad (k + 1 \leqslant i \leqslant n). \qquad (1)$$

In this octant, the points of P_n satisfy (1) and

$$x_1 + x_2 + \cdots + x_k - x_{k+1} - x_{k+2} - \cdots - x_n + |x_1 + x_2 + \cdots + x_n| \leqslant 2.$$

Equivalently (as can be easily verified by considering the two cases $\sum_{i=1}^{n} x_i \geqslant 0$ or $\leqslant 0$),

$$x_i \geqslant 0 \quad (1 \leqslant i \leqslant k) \quad \text{and} \quad x_i \leqslant 0 \quad (k+1 \leqslant i \leqslant n),$$

$$\sum_{i=1}^{k} x_i \leqslant 1 \quad \text{and} \quad \sum_{i=k+1}^{n} x_i \geqslant -1.$$

In order to compute the content of this part of P_n we let $u_i = -x_{k+i}$ $(1 \leqslant i \leqslant n-k)$ and compute the volume of the congruent polytope whose points $(x_1, x_2, \ldots, x_k, u_1, u_2, \ldots, u_{n-k})$ satisfy:

$$x_i \geqslant 0 \quad (1 \leqslant i \leqslant k) \quad \text{and} \quad u_i \geqslant 0 \quad (1 \leqslant i \leqslant n-k),$$

$$\sum_{i=1}^{k} x_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{n-k} u_i \leqslant 1.$$

This volume is given by

$$\int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-\sum_{i=1}^{k-1} x_i} \int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-\sum_{i=1}^{n-k-1} u_i} du_{n-k} \cdots du_2\, du_1\, dx_k \cdots dx_2\, dx_1$$

which (à la Hsu) evaluates to $(1/(n-k)!)(1/k!)$. Thus, the part of P_n in the $\binom{n}{k}$ octants (each with k nonnegative and $n-k$ nonpositive coordinates) is

$$\binom{n}{k} \frac{1}{k!\,(n-k)!} = \frac{1}{n!} \binom{n}{k} \binom{n}{n-k}$$

and the content of P_n is

$$\sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} \binom{n}{n-k} = \frac{1}{n!} \binom{2n}{n},$$

where this last equality is simply Vandermonde's identity. [See, for example, John Riordan's *An Introduction to Combinatorial Analysis*, Wiley and Sons (1958)15.]

The Derivatives of Sin x and Cos x

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In this note, we offer simple proofs of the formulas $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ for x acute. Both derivations rest on the figure below and avoid the necessity of first deriving $\lim_{h \to 0} \frac{\sin h}{h} = 1$. They also do not require the use of the formula for the difference of two sines or the formula for the sine of