CLASSROOM CAPSULES

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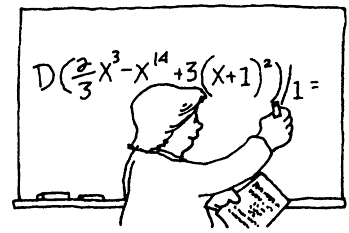

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

From Square Roots to *n*-th Roots: Newton's Method in Disguise

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The 3,000-year-old Babylonian method, expressed in the modern formula (1) below, of producing rational numbers converging rapidly to $\sqrt{2}$ is easily seen to give the same approximants as obtained by applying Newton's method to solve the equation $x^2 - 2 = 0$. Newton's method almost always yields quadratic convergence, as demonstrated in almost every book on numerical analysis. This means we get a striking accuracy of roughly twice as many decimal places in the output approximant y as in the input x, where the Babylonian output is identical to that obtained by the "divide and average" process:

$$y = \frac{1}{2}\left(x + \frac{2}{x}\right). \tag{1}$$

What is the conceptual basis of this magically efficient algorithm? One way the Babylonians (or, later, the Greeks) might have viewed it is to reflect that if a rectangle has an area of 2 then its sides have to be x and $2/x$. The average of these sides is thus closer to the side of a square of area 2 than was either side of the original rectangle (assuming the original rectangle was nearly square). If this geometric interpretation accurately reflects the conceptual basis for this efficient square-root algorithm, then there is a natural way to proceed to search for an equally efficient cube-root algorithm.

To find $\sqrt[3]{2}$, for example, we would note that if a rectangular solid—which we might naturally take to have a square base—has a volume of 2, then its sides have to be x, x, and $2/x^2$. Then we should expect the average of these *three* quantities to be closer to the side of a cube of volume 2 than was either side of the initial rectangular solid, and we immediately guess that our algorithm should give the output y defined by

$$y = \frac{1}{3}\left(x + x + \frac{2}{x^2}\right). \tag{2}$$

Now (surprise!) this algorithm is just as efficient. It produces quadratic convergence to $\sqrt[3]{2}$, because (2) happens to produce outputs identical to those obtained by applying Newton's method to solve $x^3 - 2 = 0$. If we start with an input of $5/4$, an easily discovered quotient of small integers whose cube is quite close to 2, then we successively get the approximants $5/4$, $63/50$, and $375047/297675$, the last of which is $1.259921055\ldots$ and thus agrees with $\sqrt[3]{2}$ to eight decimal places.

The pedagogical value of this is that the instructor can follow it by challenging students to figure out how to get quadratic convergence to fourth roots, fifth roots, etc. by extending this line of thought to higher dimensions unrecognized by the Greeks. With the instructor's encouragement, some students may dare to venture into the "fourth dimension" and see that to approach $\sqrt[4]{2}$, for example, they should make an approximation x and take their next approximant y to be

$$y = \frac{1}{4}\left(x + x + x + \frac{2}{x^3}\right). \tag{3}$$

Students may go on from (3) to discover, analogously, how to get the n-th root of an arbitrary positive number (and thereby effortlessly obtain the ability to handle rational roots as well, since $a^{m/n}$ is the n-th root of a^m).

Virtually any reasonable guess will yield an algorithm leading to convergence to a desired root, but the rate of convergence for uninspired guesses can be agonizingly slow. If, for example, in approximating $\sqrt[3]{2}$ one should guess that the output y should be simply the average of x and $2/x^2$, then an initial input of $5/4$ in the resulting algorithm will require about twenty iterations to get the precision obtained by two applications of formula (2).

Playing with such algorithms will lead students to realize that a mathematical task may often be done in different ways, but that in mathematics, as elsewhere, we especially value the way that is most direct or least wasteful. Our aim is not just to get the job done, but to do it with some sense of style. Introducing this playful approach before introducing derivatives enables students to experience something of the nature of mathematical research, encourages them to try it on their own, and may even get them to think about the value of what they do. Later, many of them will appreciate more deeply the deftness of Newton's method, which duplicates "their" research and does so much more.

Amortization: An Application of Calculus

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It is very easy to slight the Intermediate Value Theorem and relegate the Monotonicity Theorem to curve-sketching in first-year calculus classes. Here we present a simple application of these theorems to the amortization of a loan. This application is difficult to find in calculus or mathematics of finance texts, but we believe it is well-suited as a small project for first-year calculus students.