We invite the reader to show that the ellipse and the hyperbola are characterized by their reflection properties by investigating the following exercises.

Exercises

1. Light rays emitted from the point F_1 $(c, 0)$ $(c > 0)$ strike a plane curve C and reflect back through the point F_2 $(-c, 0)$. Assuming that the angle of incidence equals the angle of reflection, show that the differential equation $xy(dy/dx)^2 + (x^2 - c^2 - y^2)(dy/dx) - xy = 0$ describes the shape of the curve C.
2. Show that for any $a > 0$, the curve $x^2/a^2 + y^2/(a^2 - c^2) = 1$ satisfies the equation in Exercise 1. Notice that for $a > c$, the curve is an ellipse with foci F_1 $(c, 0)$ and F_2 $(-c, 0)$. And for $a < c$ the curve is a hyperbola with the same foci.
3. Discuss the uniqueness of the solutions in Exercise 2, for the differential equation in Exercise 1.

The method we used to solve the differential equation (1) does not work for the equation in Exercise 1. Can you find a way to solve it?

References

1. Bernard Banks, *Differential Equations with Graphical and Numerical Methods*, Prentice Hall, 2001.
2. R. E. Johnson, and F. L. Kiokemeister, *Calculus and Analytic Geometry* (3rd ed.), Allyn and Bacon, 1964.

Sums of Roots and Poles of Rational Functions

Paul Deiermann (pdeiermann@semovm.semo.edu), Southeast Missouri State University, Cape Girardeau, MO 63701

Consider rational functions $\frac{p(x)}{q(x)}$ expressed as $\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$. For the examples

$$\frac{x^3 + 3x^2}{x^2 - x - 2} = x + 4 + \frac{6x + 8}{x^2 - x - 2} \quad \text{and} \quad \frac{6x^2 - 5}{2x + 1} = 3x - \frac{3}{2} - \frac{7/2}{2x + 1},$$

we see that the sum of Q's roots equals the sum of p's roots minus the sum of q's roots. This illustrates an interesting tidbit concerning rational functions that probably falls under the category of a known, but not well-known, fact among those of us in the trenches.

Let $p(x)$ and $q(x)$ be polynomials with no common factors and with real coefficients, where the respective degrees of p and q are n and m with $n > m \geq 0$. Using polynomial long division, we can write $\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$ with the degree of $R(x)$ strictly less than the degree of $q(x)$. Then the sum of Q's roots equals the sum of p's roots minus the sum of q's roots.

Proof. Since we are interested only in the roots and poles, we may assume that the leading coefficients of $p(x)$ and $q(x)$ (and hence of $Q(x)$ also) are unity. Let

$$p(x) = \prod_{k=1}^{n}(x - a_k) = x^n - \left(\sum_{k=1}^{n} a_k\right) x^{n-1} + O(x^{n-2}) = x^n - Ax^{n-1} + O(x^{n-2}),$$

$$q(x) = \prod_{k=1}^{m}(x - b_k) = x^m - \left(\sum_{k=1}^{m} b_k\right) x^{m-1} + O(x^{m-2}) = x^m - Bx^{m-1} + O(x^{m-2}),$$

$$Q(x) = \prod_{k=1}^{n-m}(x - c_k) = x^{n-m} - \left(\sum_{k=1}^{n-m} c_k\right) x^{n-m-1} + O(x^{n-m-2})$$
$$= x^{n-m} - Cx^{n-m-1} + O(x^{n-m-2}).$$

Dividing $p(x)$ by $q(x)$, we see that $\frac{p(x)}{q(x)} = x^{n-m} - (A - B)x^{n-m-1} + O(x^{n-m-2})$. By comparing coefficients, it is obvious that $C = A - B$.

Example: Let $\frac{p(x)}{q(x)} = \frac{(x+1)(x-3)^2(x+4)^3}{(x-7)(x+2)^3}$. Long division reveals that $Q(x) = x^2 + 8x + 29$, which has roots $-4 \pm i\sqrt{13}$. Thus, the sum of Q's roots is -8, which equals the sum of p's roots minus the sum of q's roots.

Learning Linear Algebra

Tommy Dreyfus (tommy.dreyfus@weizmann.ac.il) of the Center for Technological Education gives some additional references to go with the paper "Teaching Linear Algebra" by Dan Kalman and Jane Day that appeared in the May 2001 *Journal*, pages 162–168, notably a book,

On the Teaching of Linear Algebra, by J.-L. Dorier, Kluwer, 2000.

Among papers not native to the United States are

A. Sierpinska, T. Dreyfus, and J. Hillel, Evaluation of a teaching design in linear algebra, *Recherche en Didactique des Mathématiques* **19** (1991) #1, 7–40.

J. Hillel and A Sierpinska, On one persistent mistake in linear algebra, in *Proceedings of the 18th International Conference of the International Group for the Pyschology of Mathematics Education*, edited by J. Pedro da Ponte and J. Matos, vol. III, 65–72, Lisbon, 1994.

M. Dias and M. Artigue, Articulation problems between different systems of symbolic representations in linear algebra, in *Proceedings of the 19th Annual Meeting of the International Group for the Pyschology of Mathematics Education*, edited by L. Meira and D. Carraher, vol. II, 34–41, Universidade Federal de Pernambuco, Recife, 1995.