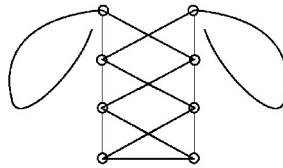


AIME PRACTICE QUESTIONS

Problem

The 8 eyelets for the lace of a sneaker all lie on a rectangle, four equally spaced on each of the longer sides. The rectangle has a width of 50 mm and a length of 80 mm. There is one eyelet at each vertex of the rectangle. The lace itself must pass between the vertex eyelets along a width side of the rectangle and then crisscross between successive eyelets until it reaches the two eyelets at the other width side of the rectangle as shown. After passing through these final eyelets, each of the ends of the lace must extend at least 200 mm farther to allow a knot to be tied. Find the minimum length of the lace in millimeters.



2014 AIME I, Problem #1—

Hint

“Find triangles and calculate the hypotenuse.”

Solution

Answer (790): The lace must be long enough to pass along one width of the rectangle and six diagonal crisscrosses, and include the two loose ends for tying the knot. The width is 50 mm. The diagonals are hypotenuses of right triangles with legs measuring $\frac{80}{3}$ mm and 50 mm. Because $50 = \frac{150}{3}$, these numbers are proportional to the first two terms of the Pythagorean triple 8, 15, 17, and the diagonal length is $\frac{170}{3}$ mm. The total length required is therefore $50 + 6 \cdot \frac{170}{3} + 2 \cdot 200 = 790$ mm.

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 2. Reason abstractly and quantitatively; 4. Model with mathematics; 8. Look for and express regularity in repeated reasoning.

CCSS-M: G-SRT-C. Define trigonometric ratios and solve problems involving right triangles.

Problem Worksheets, continued

Problem

An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N .

2014 AIME I, Problem #2—

Hint

“Consider having the same color as the union of two disjoint events. Calculate the probability of each event.”

Solution

Answer (144):

The event “both balls have the same color” is the union of two disjoint events, “ball 1 and ball 2 are both green” and “ball 1 and ball 2 are both blue.” Because the selections from the two urns are independent, the probability that the two balls are the same color is

$$\begin{aligned} P(\text{ball 1 green}) \cdot P(\text{ball 2 green}) + P(\text{ball 1 blue}) \cdot P(\text{ball 2 blue}) \\ = \frac{4}{10} \cdot \frac{16}{16+N} + \frac{6}{10} \cdot \frac{N}{16+N} \\ = .58. \end{aligned}$$

Multiplying by $100(16+N)$ yields $40 \cdot 16 + 60 \cdot N = 58(16+N)$ which reduces to $N = 144$.

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 2. Reason abstractly and quantitatively; 4. Model with mathematics; 8. Look for and express regularity in repeated reasoning.

CCSS-M: S-CP.B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Problem Worksheets, continued

Problem

Find the number of rational numbers r , $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and denominator have a sum of 1000.

2014 AIME I, Problem #3—

Hint

“Show why the numerator is relatively prime to 1000.”

Solution

Answer (200):

Let $r = \frac{a}{b}$ be a rational number in lowest terms with $a + b = 1000$. If $\gcd(a, 1000) = d > 1$, then d is also a factor of $b = 1000 - a$ so $\frac{a}{b}$ is not in lowest terms. On the other hand if $\gcd(a, 1000) = 1$ and $b = 1000 - a$, then $\gcd(a, b) = 1$, and $\frac{a}{b}$ is in lowest terms. Because $1000 = 2^3 \cdot 5^3$, the number of positive integers less than 1000 and relatively prime to 1000 is

$$\phi(1000) = 1000 - \frac{1000}{2} - \frac{1000}{5} + \frac{1000}{10} = 400.$$

Half of these numbers (i.e. 200) are less than $\frac{1}{2} \cdot 1000$, and these are the possible numerators for a fraction of the desired type.

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 2. Reason abstractly and quantitatively; 8. Look for and express regularity in repeated reasoning.

CCSS-M: N-RN-B. Use properties of rational and irrational numbers.

Problem Worksheets, continued

Problem

Jon and Steve ride their bicycles along a path that parallels two side-by-side train tracks running in the east/west direction. Jon rides east at 20 miles per hour, and Steve rides west at 20 miles per hour. Two trains of equal length, traveling in opposite directions at constant but different speeds, each pass the two riders. Each train takes exactly 1 minute to go past Jon. The westbound train takes 10 times as long as the eastbound train to go past Steve. The length of each train is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m + n$.

2014 AIME I, Problem #4—

Hint

‘Express the speed of the trains relative to Jon, then use this to express the speeds relative to Steve.’

Solution

Answer (049):

Let the length of each train be L miles. In passing each rider, each train travels L miles relative to that rider. Because the trains each go past Jon in 1 minute, their speed relative to Jon is L miles per minute. Jon and Steve are each riding at a speed of $\frac{1}{3}$ mile per minute in opposite directions. Therefore, relative to Steve, the speed of the eastbound train is $L + \frac{2}{3}$ miles per minute, and the speed of the westbound train is $L - \frac{2}{3}$ miles per minute. The times required for the trains to go past Steve are $\frac{L}{L + \frac{2}{3}} = \frac{3L}{3L + 2}$ and $\frac{L}{L - \frac{2}{3}} = \frac{3L}{3L - 2}$, respectively. Thus $\frac{3L}{3L - 2} = 10 \left(\frac{3L}{3L + 2} \right)$, from which $L = \frac{22}{27}$. The requested sum is $22 + 27 = 49$.

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 2. Reason abstractly and quantitatively; 4. Model with mathematics; 8. Look for and express regularity in repeated reasoning.

CCSS-M: A-CED.A Create equations that describe numbers or relationships.

Problem Worksheets, continued

Problem

Let the set $S = \{P_1, P_2, \dots, P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called communal if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset.)

2014 AIME I, Problem #5—

Hint

“Characterize communal sets in terms of arcs of the circle, and consider rotations.”

Solution

Answer (134):

It is clear that S and the empty set are both communal. Let C be the circle that passes through the points of S . A proper subset Q of S is communal if and only if there is an arc of C such that Q is the set of points of S that lie on that arc. To see this let Q be a communal subset of S , and let D be a circle so that the points of Q are inside of D , and the points of S not in Q are outside of D . Because the portion of C that is inside of D is an arc, the points of Q must be the points of S that lie on this arc.

Now let Q be the set of points in S that lie on an arc α of C . If necessary, extend α slightly so that no points of S are endpoints of α . Let A and B be the endpoints of α , and let M be the midpoint of α . Let D be the circle with center M passing through A and B . Then α is the arc of C inside of D , all points of Q are inside of D , and all points of S not in Q are outside of D . This shows that Q is communal.

Let $1 \leq k \leq 11$, and let Q be a communal subset with k points. Then each rotation of Q through an angle of $\frac{360}{12} = 30^\circ$ results in another communal subset of k elements. With successive rotations this process generates 12 communal subsets of k elements. Thus the number of communal subsets is $11 \cdot 12 + 2 = 134$.

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.

CCSS-M: None.