and $J_n = \{x_0 \in I \mid |f(x_0^+) - f(x_0^-)| > 1/n\}$. Show that $R \cup J = \bigcup_{n=1}^{\infty} S_n$ and use Lemma 2 to prove that each S_n is countable. Now use the fact that the union of a countable collection of countable sets is countable to establish the following:

THEOREM. *Let f be a real-valued function (not necessarily bounded) defined on any interval. Then $R \cup J$ is countable.*

REFERENCES

1. Nicholas J. DeLillo, *Advanced Calculus with Applications*, Macmillan, New York, 1982.
2. William F. Trench, *Advanced Calculus*, Harper and Row, New York, 1978.
3. Walter Rudin, *Principles of Mathematical Analysis*, third edition, McGraw-Hill, New York, 1976, p. 100, #17.
4. Karl R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Belmont, California, 1981, p. 131, #3, p. 280, #24.

Rings on R^2

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A common example of a multiplication of ordered pairs is afforded by the complex numbers when we identify the ordered pair of real numbers (a, b) with $a + bi$. The multiplication is $(a, b) * (c, d) = (ac - bd, ad + bc)$. Another example is provided by the rational numbers extended by the square root of 2. If we identify the ordered pair of rational numbers (a, b) with $a + b\sqrt{2}$ we have $(a, b) * (c, d) = (ac + 2bd, ad + bc)$.

How many ways can we define a multiplication of ordered pairs which will result in a ring and which of these rings will be fields? Can we generalize our results to analyze the multiplication of ordered triples?

The intent of this note is to address these questions using the basic concepts of linear algebra. We will construct all the multiplications for ordered pairs of real numbers and give an elementary proof of the fact that there is no three-dimensional analogue for the complex numbers.

Notation: R^2 and R^3 will denote the vector spaces on $R \times R$ and $R \times R \times R$ over the field R of *real numbers*.

Let $N[R^2]$ be any ring defined on $R \times R$ with identity $(1, 0)$. Let $X, Y, Z \in N[R^2]$ and $r \in R$. We will assume $N[R^2]$ satisfies the following conditions in addition to the axioms for a ring.

a) Addition in $N[R^2]$ is vector addition in R^2.

b) The real numbers are imbedded as ordered pairs of the form $(r, 0)$ and $(r, 0) * X = X * (r, 0)$ for all $X \in N[R^2]$ and $r \in R$.

c) Scalar multiplication in the vector space R^2 is identified with multiplication in

$N[R^2]$ as follows: $rX = (r,0) * X$. Since $(a,b) = a(1,0) + b(0,1)$ and $(c,d) = c(1,0) + d(0,1)$, multiplication in $N[R^2]$ is uniquely determined by the product of $(0,1)$ with itself. If $(0,1)*(0,1) = (p,q)$ then the multiplication is given by

$$(a,b)*(c,d) = (ac + pbd, ad + bc + qbd). \tag{1}$$

What choices of p and q will satisfy the ring axioms for multiplication? It is not immediately clear that they can be chosen independently. There may be some constraints imposed by associativity, however, we shall soon show that any choice of p and q produces an associative multiplication. We will denote the ring $N[R^2]$ with this multiplication by $N[R^2, p, q]$. The ring of complex numbers is $N[R^2, -1, 0]$.

If $X \in N[R^2, p, q]$ does X have a multiplicative inverse? Is X a divisor of zero? If each X in $N[R^2, p, q]$ could be represented by a matrix these would be easy questions to answer. So, we will proceed to associate a 2×2 matrix with each element of $N[R^2, p, q]$ by observing the following:

$$X*((r,0)*Y) = (r,0)*(X*Y) \quad \text{and} \quad X*(Y+Z) = X*Y + X*Z. \tag{2}$$

The first equation follows from the associativity of $*$ and the commutativity of scalars. The second is the distributive law of the ring. Next we recall that a mapping F from R^2 to R^2 is linear if

$$F(rY) = rF(Y) \quad \text{and} \quad F(Y+Z) = F(Y) + F(Z). \tag{3}$$

Comparing (2) and (3) we see that left multiplication by any X in the ring $N[R^2, p, q]$ is a linear transformation of the vector space R^2! Thus we can associate a 2×2 matrix with each element of $N[R^2, p, q]$. We will denote this matrix by $M(X)$. If we treat ordered pairs as column vectors then $X * Y$ will be computed as $M(X)Y^T$ and the columns of $M(X)$ will be the images of $(1,0)$ and $(0,1)$, namely, $X*(1,0)$ and $X*(0,1)$. The matrices which correspond to $(1,0)$ and $(0,1)$ are given by:

$$M(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M(0,1) = \begin{pmatrix} 0 & p \\ 1 & q \end{pmatrix}.$$

It is easy to verify that $M(X+Y) = M(X) + M(Y)$ and $M(rX) = rM(X)$ by examining these matrix equations column by column. The first column of $M(X+Y)$ is $(X+Y)*(1,0)$ and the first column of $M(X) + M(Y)$ is $X*(1,0) + Y*(1,0)$. The verification is easily completed in this manner and thus:

$$M(X) = M(a(1,0) + b(0,1)) = \begin{pmatrix} a & pb \\ b & a+qb \end{pmatrix}.$$

If $X = (a,b)$ and $Y = (c,d)$ then $X * Y$ is computed as follows.

$$X*Y = \begin{pmatrix} a & pb \\ b & a+qb \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac + pbd \\ ad + bc + qbd \end{pmatrix}. \tag{4}$$

Thus, we see that X is a left zero divisor iff the kernel of $M(X)$ is nontrivial and X has a right inverse iff $M(X)$ is invertible.

Let $S = \{M(X) | X \in N[R^2, p, q]\}$. It is easy to see that S is closed under matrix addition. It is somewhat surprising that S is also closed under matrix multiplication for any choice of p and q *since a direct computation shows that* $M(X)M(Y) = M(X*Y)$. Hence, S forms a subring of the ring of 2×2 matrices. We will denote this subring by $M[R^2, p, q]$. *Next we establish an isomorphism between* $N[R^2, p, q]$ *and the ring of*

matrices $M[R^2, p, q]$. Since matrix multiplication is associative this isomorphism will induce associativity in $N[R^2, p, q]$ for all p and q.

THEOREM 1. *$M[R^2, p, q]$ is isomorphic to $N[R^2, p, q]$.*

Proof. Define $M: N[R^2, p, q] \to M[R^2, p, q]$ by the matrix $M(X)$ defined as above. Since the first column of $M(X)$ is X itself, it is easy to see that M is $1:1$ and onto. *Note that the second column is uniquely determined by the values of a and b in the first column and the choice of p and q.*

$M(X+Y) = M(X) + M(Y)$ and $M(X * Y) = M(X) * M(Y)$ are verified directly by matrix arithmetic.

The multiplication in $N[R^2, p, q]$ given by (1) is commutative by symmetry and thus $M[R^2, p, q]$ is also commutative. We can confirm this by verifying that, for fixed p and q, all the matrices in $M[R^2, p, q]$ have the same eigenvectors, namely,

$$\left(p, \left(q + \sqrt{q^2 + 4p}\right)/2\right) \quad \text{and} \quad \left(p, \left(q - \sqrt{q^2 + 4p}\right)/2\right). \tag{5}$$

THEOREM 2. *$N[R^2, p, q]$ is a field iff $q^2 + 4p < 0$.*

Proof. Let $X = (a, b)$ and $X \neq (0,0)$. $\mathrm{Det}(M(X)) = a^2 + qab - pb^2$. $\mathrm{Det}(M(X)) \neq 0$ iff $q^2 + 4p < 0$.

If X has an inverse then $M(X^{-1})$ must be $M(X)^{-1}$. If $q^2 + 4p \geqslant 0$ and p and q are not both zero then there exists an eigenvector Y (see (5)) such that $M(X)Y = kY = (kI)Y$ and even right cancellation fails.

When $\mathrm{Det}(M(X))$ is a quadratic form which is never zero we are able to define a field for ordered pairs of reals. If $\mathrm{Det}(M(X))$ were a cubic then there would always be at least one real zero and hence a zero divisor. With this in mind, we proceed to construct a subring of the ring of 3×3 real matrices for any ring of ordered triples of reals.

Let $N[R^3]$ be any ring defined on $R \times R \times R$ with identity $(1,0,0)$. We will assume the following.

a) Addition in N is the addition of the vector space R^3.

b) The real numbers are imbedded as ordered triples of the form $(r,0,0)$ and $(r,0,0) * X = X * (r,0,0)$ for all $X \in R^3$ and $r \in R^3$.

c) Scalar multiplication in the vector space R^3 is identified with multiplication in N as follows: $rX = (r,0,0) * X$.

The multiplication in $N[R^3]$ is completely determined by the products of the elements in the canonical basis. We will assume the following multiplication table.

$*$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(1,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(0,1,0)$	$(0,1,0)$	(p_1, p_2, p_3)	(q_1, q_2, q_3)
$(0,0,1)$	$(0,0,1)$	(r_1, r_2, r_3)	(s_1, s_2, s_3)

We note that it is no longer possible to make arbitrary choices for all the above column vectors and still have a multiplication which satisfies the associative law. *If associativity holds then it imposes restrictions on the products of* $(0,1,0)$ *and* $(0,0,1)$. *For example, if p_3 and s_2 are nonzero then $r_1 = q_1$, $r_2 = q_2$, and $r_3 = q_3$ as the reader can verify by examining the cubes of* $(0,1,0)$ *and* $(0,0,1)$.

We can again interpret left multiplication as a linear transformation and we can associate a matrix $M(X)$ with each ordered triple X.

We will treat ordered triples as column vectors and compute $X * Y$ as $M(X)Y$. The matrices of the canonical basis elements are shown below.

$$\begin{array}{lccc} X: & (1,\ 0,\ 0) & (0,\ 1,\ 0) & (0,\ 0,\ 1) \\ M(X): & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & p_1 & q_1 \\ 1 & p_2 & q_2 \\ 0 & p_3 & q_3 \end{pmatrix} & \begin{pmatrix} 0 & r_1 & s_1 \\ 0 & r_2 & s_2 \\ 1 & r_3 & s_3 \end{pmatrix}. \end{array}$$

We can still show $M(X+Y) = M(X) + M(Y)$ and $M(rX) = rM(X)$ for any choice of the p, q, r, and s vectors as we did earlier with $N[R^2]$.

If $X = (a, b, c)$ then

$$M(X) = \begin{pmatrix} a & bp_1 + cr_1 & bq_1 + cs_1 \\ b & a + bp_2 + cr_2 & bq_2 + cs_2 \\ c & bp_3 + cr_3 & a + bq_3 + cs_3 \end{pmatrix}$$

and

$$\begin{aligned} \operatorname{Det}(M(X)) = {} & a^3 + (p_2 + q_3)a^2b + (r_2 + s_3)a^2c + (p_2q_3 - p_3q_2 - p_1)ab^2 \\ & + (r_2s_3 - r_3s_2 - s_1)ac^2 + (p_2s_3 - p_3s_2 - q_1 - r_1)abc \\ & + (r_3s_1 - r_1s_3 + p_1s_2 - p_2s_1)bc^2 + (p_3s_1 - p_1s_3 + p_1q_2 - p_2q_1)b^2c \\ & + (p_3q_1 - p_1q_3)b^3 + (r_1s_2 - r_2s_1)c^3. \end{aligned}$$

Using theorem 1 we took advantage of associativity in $M[R^2]$ to establish associativity in $N[R^2]$. This time choose the products of the basis elements in $N[R^3]$ so that multiplication is associative in $N[R^3]$ and construct the isomorphic subring of 3×3 matrices. Now the value of the isomorphism comes from the determinant in $M[R^3]$ which can be used to detect the presence of zero divisors back in $N[R^3]$.

If $\operatorname{Det}(M(X)) = 0$ then there exists a nonzero Y such that $M(X)Y = (0,0,0)$ and thus $X * Y = 0$. If we choose $X = (x, 1, 0)$ then $\operatorname{Det}(M(X))$ is a monic cubic in x which has at least one real root and hence there is always a zero divisor of the form $(x, 1, 0)$ which establishes the fact that there are no fields on R^3.

REFERENCES

The problem of constructing a multiplication for a vector space is discussed in the literature under the subject heading of hypercomplex numbers or linear associative algebra. A good narrative description of the development of this topic can be found in M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972, Chapter 32, and E. Kramer, *The Nature and Growth of Modern Mathematics*, Hawthorn Books, 1970, Chapter 28.

The search initiated by Hamilton and Grassmann for a three-dimensional analogue of the complex numbers was proven to be futile by Frobenius in 1878. A nice presentation of this theorem of Frobenius appears in I. Herstein, *Topics in Algebra*, Blaisdell, 1964, Chapter 7.

A brief introduction to hypercomplex numbers may be found in B. van der Waerden, *Modern Algebra*, Vol. I, Frederick Ungar, 1953. References to a more complete treatment of the subject can be found in the bibliography at the end of chapter 32 in the text by Kline.

Further material of historical and technical interest appears in M. Crowe, *A History of Vector Analysis*, University of Notre Dame Press, 1967, which has recently become available as a Dover reprint.