Thinning Out the Harmonic Series

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Introduction It is well known that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \tag{1}$$

is divergent, and its partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

is close to $\log n$; this shows that the harmonic series diverges very, very slowly. For example, it takes more than 1.5×10^{43} terms for its partial sums to reach 100; see **[3]**, **[4]**, and **[5]**. In this paper, we refine and thin out the harmonic series to show that the divergence of this series depends on some of its specific terms and without those terms the remaining subseries is convergent. Then, an upper bound and an approximate value will be given to the value of the convergent subseries. Finally, by use of the derived results, it will be shown that the divergence of the Euler series $\sum \frac{1}{p}$ (over prime numbers p) depends on specific prime numbers.

We begin with two interesting results about subseries of (1). For simplicity, it is convenient to introduce a little terminology here. Let $r = 0, 1, 2, \ldots, 9$. We say that the positive integer n *contains* r, if r is one of the digits of n, *i.e.*, the decimal representation of n contains at least one r, and we write $r \in n$; otherwise n is called an $r-$*free* integer and we will denote this by $r \notin n$. For example, 16 contains 1 and 6, but it is obviously a $2-$free number, and we can also write $2 \in 289$, $5 \in 1059$, and $6 \notin 281$.

PROPOSITION 1. *The subseries*

$$\sum_{9 \in n} \frac{1}{n} = \frac{1}{9} + \frac{1}{19} + \frac{1}{29} + \cdots + \frac{1}{89} + \frac{1}{90} + \frac{1}{91} + \cdots + \frac{1}{99} + \cdots, \tag{2}$$

of the harmonic series (1), *which consists of all terms of* (1) *with one or more nines, is a divergent series.*

Proof. It is clear that the series (2) with positive terms is greater than the divergent series

$$\sum_{k=0}^{\infty} \frac{1}{10k+9},$$

and so it is a divergent series.

Now, if all the terms of the subseries (2) are deleted (omitted) from the original harmonic series (1), the remaining series

$$\sum_{9 \notin n} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} + \frac{1}{10} + \cdots, \tag{3}$$

consisting of all $9-$free terms of (1), appears to be divergent. If we compare the terms of the two series (2) and (3), we see that, between 1 and 9 there is only one

term from (2) but there are eight terms from (3), and similarly between 10 and 99, there are 18 terms from (2), but 72 terms from (3), and so on. For these reasons, it seems that the series (3) is divergent. In fact, though, the series (3) is convergent, and this has been known for a long time! In 1914, Kempner **[8]** proved this by using mathematical induction, and he called (3) "A Curious Convergent Series"; see also **[1]**, **[2]**, **[3]**, **[7]**, **[9]**, **[10]**, **[12]** and **[13]**. Here, we will give a different proof, which gives us a sharper upper bound for the sum of (3). Then by extending this idea we will find further interesting and new surprising results about other subseries of the harmonic series (1).

PROPOSITION 2. *The series* (3) *is convergent.*

Proof. There are only 8×9^k different $(k+1)$-digit positive integers with their digits belonging to $\{0, 1, 2, \ldots, 8\}$. So, for any natural number k, there are only 8×9^k 9-free positive integers between 10^k and $10^{k+1} - 1$. The series (3) can be rearranged and rewritten as:

$$\begin{aligned}
\sum_{9 \notin n} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} + \frac{1}{10} + \cdots \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{10} + \cdots + \frac{1}{88}\right) \\
&\quad + \left(\frac{1}{100} + \cdots + \frac{1}{888}\right) + \cdots + \left(\frac{1}{10^k} + \cdots + \frac{1}{\underbrace{88\cdots8}_{k \text{ times}}}\right) + \cdots \\
&< \left(\overbrace{1 + 1 + \cdots + 1}^{8 \times 9^0 \text{ times}}\right) + \left(\overbrace{\frac{1}{10} + \frac{1}{10} + \cdots + \frac{1}{10}}^{8 \times 9^1 \text{ times}}\right) + \cdots + \left(\overbrace{\frac{1}{10^k} + \cdots + \frac{1}{10^k}}^{8 \times 9^k \text{ times}}\right) + \cdots \\
&= \left(8 \times 9^0 \times \frac{1}{10^0}\right) + \left(8 \times 9^1 \times \frac{1}{10^1}\right) + \left(8 \times 9^2 \times \frac{1}{10^2}\right) \\
&\quad + \cdots + \left(8 \times 9^k \times \frac{1}{10^k}\right) + \cdots = 8 \sum_{k=0}^{\infty} \left(\frac{9}{10}\right)^k = 80.
\end{aligned}$$

This proves that (3) converges.

For any positive integer k, let N_k be the number of all 9-free positive integers that are less than 10^k, and M_k be the number of positive integers that are less than 10^k and contain at least one digit 9. From the above proof we have $N_k = \sum_{i=0}^{k-1} 8 \times 9^i = 9^k - 1$ and $M_k = (10^k - 1) - (9^k - 1) = 10^k - 9^k$. These results imply that $\lim_{k \to \infty} N_k / M_k = 0$.

The series (3) converges very, very slowly. The value of its partial sum with 10^5 terms reaches 12.0908, which is relatively very far from the actual value of the series, 22.921. After 10^5 terms from this convergent series, the partial sum and the exact value do not have even a single digit in common!

Obviously, the digit 9 does not play a critical role in Propositions 1 and 2, so we have the following theorem.

THEOREM 1. *For any number* $r = 0, 1, \ldots, 8, 9$,

$$\text{(I) the series } \sum_{r \in n} \frac{1}{n} \text{ is divergent,} \tag{4}$$

$$\text{(II) the series } \sum_{r \notin n} \frac{1}{n} \text{ is convergent.} \tag{5}$$

Recently, Baillie [2] has calculated (to 20 decimal places) the value of the convergent series (5) for $r = 0, 1, \ldots, 9$. Table 1 gives three decimal-place values of these sums $T_0, T_1, T_2, \ldots, T_9$; see also [3] and [11].

TABLE 1

$T_0 = 23.103$	$T_1 = 16.177$	$T_2 = 19.257$	$T_3 = 20.570$	$T_4 = 21.327$
$T_5 = 21.835$	$T_6 = 22.206$	$T_7 = 22.493$	$T_8 = 22.726$	$T_9 = 22.921$

We can extend the above result by changing the base from base 10 to any particular base b. For example, if we consider base 100 then we obtain a convergent series by deleting any particular digit in base 100; see [3]. Also, if we look at Table 1, we see that the values of the T_i are increasing for $i \geq 1$. This suggests the following *open question* (prove or disprove).

Open Question. *For any base b, is $T_m < T_n$, whenever $1 \leq m < n$?*

New surprising results Let us consider two special cases in Theorem 1, with $r = 9$ and $r = 8$:

$$\text{(I) the series } \sum_{9 \in n} \frac{1}{n} \text{ is divergent,} \tag{6}$$

$$\text{(II) the series } \sum_{8 \notin n} \frac{1}{n} \text{ is convergent.} \tag{7}$$

It is obvious that the convergent series (7) contains all terms of the divergent series (6), except those terms that contain both digits 8 and 9 (like $89, 98, 189, \cdots$). So, the divergence of (6) and also the harmonic series (1) depends on such terms. This means that the series

$$\sum_{8,9 \in n} \frac{1}{n} = \frac{1}{89} + \frac{1}{98} - \frac{1}{189} + \cdots \tag{8}$$

is divergent, and its complement with respect to (1) is convergent. In general, we have the following theorem.

THEOREM 2. *For any two different integers $r, s = 0, 1, \ldots, 9$,*

$$\text{(I) } \textit{the series } \sum_{r,s \in n} \frac{1}{n} \textit{ is divergent,} \tag{9}$$

$$\text{(II) } \textit{the series } \sum_{r \notin n \vee s \notin n} \frac{1}{n} \textit{ is convergent.} \tag{10}$$

If we sum the 10 convergent series with their sums given in Table 1, then the resulting series

$$T = \sum_{0 \notin n} \frac{1}{n} + \sum_{1 \notin n} \frac{1}{n} + \cdots + \sum_{9 \notin n} \frac{1}{n} \tag{11}$$

will be a convergent series with sum less than 213. Obviously, there are still some other terms in the harmonic series that do not belong to the convergent series (11). All such terms contain each of the digits $0, 1, 2, \ldots, 9$, at least once, and $\frac{1}{1023456789}$ is the first of these terms. Now we state the main theorems of this paper.

THEOREM 3. *Let D be the set of positive integers that contain each of the digits $0, 1, \ldots, 9$ at least once; then*

$$\text{(I) the series } \sum_{n \in D} \frac{1}{n} \text{ is divergent,} \tag{12}$$

$$\text{(II) the series } \sum_{n \notin D} \frac{1}{n} \text{ is convergent.} \tag{13}$$

THEOREM 4. *For any positive integer k, let N_k be the number of positive integers n that are less than 10^k and $n \in D$, and M_k be the number of positive integers m that are less than 10^k and $m \notin D$. Then $\lim_{k \to \infty} N_k/M_k = 0$.*

Proof. This theorem is a special case of the following theorem.

THEOREM 5. *Suppose that C is a subset of positive integers and $\sum_{n \in C} 1/n$ is convergent. For any positive integer k, let N_k be the number of elements in C that are $\leq 10^k$, and $M_k = 10^k - N_k$. Then $\lim_{k \to \infty} N_k/M_k = 0$.*

Proof. It suffices to show that

$$\lim_{k \to \infty} \frac{N_k}{10^k} = 0.$$

Since $N_k - N_{k-1}$ is the number of elements of $C_k = \{n \in C \mid 10^{k-1} < n \leq 10^k\}$, we have

$$\sum_{n \in C_k} \frac{1}{n} \geq \sum_{n \in C_k} \frac{1}{10^k} = \frac{N_k - N_{k-1}}{10^k}$$

and so

$$\sum_{k=2}^{\infty} \frac{N_k - N_{k-1}}{10^k} \leq \sum_{n \in C} \frac{1}{n} < \infty.$$

Thus

$$\lim_{k \to \infty} \frac{N_k - N_{k-1}}{10^k} = 0.$$

If we set $a_k = N_k/10^k$, then for all k, we have $a_k \leq 1$ and $\lim_{k \to \infty}(a_k - a_{k-1})/10 = 0$. Now we show that $\lim_{k \to \infty} a_k = 0$. Let $\epsilon > 0$. There exists an integer n, such that $(a_k - a_{k-1})/10 < \epsilon$, for all $k \geq n$. By induction on m, it can be shown that for any $m \geq 1$,

$$a_{n+m} \leq \epsilon \left(\sum_{k=0}^{m-1} \frac{1}{10^k} \right) + \frac{1}{10^m}.$$

Letting $m \to \infty$, this will imply that $\limsup a_k \leq (10/9)\epsilon$, and since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{k \to \infty} a_k = 0$.

A note on the Euler series Now we consider the divergent Euler series $\sum \frac{1}{p}$, over prime numbers p (see, Hardy and Wright; [**6**; pp. 17, 120, and 349]). Again, the sum of ten convergent series (11) contains all the terms of $\sum \frac{1}{p}$, except for those prime numbers that belong to the set D. Hence the divergence of the Euler series $\sum \frac{1}{p}$ is due to such prime numbers. For example, 1012345789, 10123465789, and 10123465897 are the first three such primes. When these primes are deleted from the Euler series $\sum \frac{1}{p}$, the remaining series is convergent.

THEOREM 6. *Let E be the set of all primes that contain each of the digits* $0, 1, \ldots, 9$ *at least once. Then*

$$\text{(I) the series } \sum_{p \in E} \frac{1}{p} \text{ is divergent,} \tag{14}$$

$$\text{(II) the series } \sum_{p \notin E} \frac{1}{p} \text{ is convergent.} \tag{15}$$

The divergence of series (14) implies that the set E is infinite.

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