Acknowledgment. We thank Tamas Lengyel (of Occidental College) and the referees for many helpful suggestions. Naimi expresses gratitude to Caltech for its hospitality while he worked there on this paper during a sabbatical leave. Pelayo wishes to thank the California Alliance for Minority Participation and the Undergraduate Research Center at Occidental College for supporting this work. Theorem 4 was obtained by Radoslav Kirov and Naimi as part of a summer undergraduate research program at Occidental College.

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Why Euclidean Area Measure Fails in the Noneuclidean Plane

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One of the central and most interesting themes of noneuclidean (hyperbolic) geometry concerns the angle sum and area of polygons. A triangle—so we learn—has an angle sum of less than π, a quadrilateral one of less than 2π, and generally a n-gon one of less than $n \cdot \pi - 2\pi$. More specifically, one can establish that the number n of vertices of a polygon does not determine its angle sum, which can be anything between 0 and $n \cdot \pi - 2\pi$. This prepares the way for the remarkable conclusion that the *defect* of a n-gon, the difference between its angle sum and $n \cdot \pi - 2\pi$, has all the desired properties of an area measure.

But where does this leave conventional Euclidean area measure with the formula (base $\times$ altitude)/2 for triangles? Is the defect simply a convenient alternative for measuring area in noneuclidean geometry, or do we have to use it because the Euclidean area measure is not applicable? The second is true, and the reason is that an indispensable but often neglected property of the formula for the area measure of a triangle in a Euclidean plane cannot be carried over to the noneuclidean plane. What we refer to is the *well-definedness* of Euclidean area measure, in particular the fact that in a triangle with sides a, b, c and related altitudes h_a, h_b, h_c, the area can be calculated as

$$\frac{ah_a}{2}, \frac{bh_b}{2}, \frac{ch_c}{2}$$

with the results in all three cases invariably being equal.

For the investigation of the same formula on the noneuclidean side, we choose a path that contrasts the two theories clearly, and which can be taken before or after one discusses the defect.

We first turn to a figure consisting of a triangle $\triangle OAB$ with a right angle at B, and points A' on ray $\overrightarrow{OA}$ such that $\overrightarrow{OA'} = 2 \cdot \overrightarrow{OA}$, and B' on ray $\overrightarrow{OB}$ such that $\triangle OA'B'$ is a triangle with right angle at B', as in FIGURE 1. In addition we define A'' as the point on ray $\overrightarrow{B'A'}$ that satisfies $BA \equiv B'A''$, and B'' as the image of A'' under the translation $[A' \to A] = [A \to O]$. It follows that $\Delta AA'A'' \equiv \Delta OAB''$. (Note: Readers may be unfamiliar with translations along a line l in noneuclidean geometry; they can be easily explained in terms of reflections. To construct the translation $[A \to O]$, create line r perpendicular to $\overrightarrow{OA}$ through A and line s perpendicular to $\overrightarrow{OA}$ through the midpoint of OA; reflecting through r and then through s produces the desired translation. [**3**, p. 326])

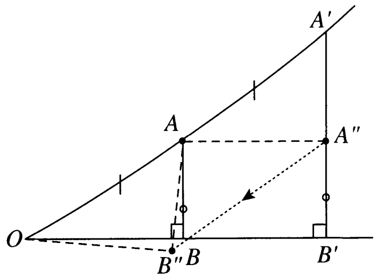

Figure 1

Remembering that triangles have angle sums $< \pi$ and quadrilaterals have angle sums $< 2\pi$, we recognize that in triangle ΔAOB, $\angle AOB + \angle OAB < \pi/2$, and in quadrilateral $BB'A''A$, $\angle BAA'' = \angle B'A''A < \pi/2$. Consequently,

$$\angle AOB < \frac{\pi}{2} - \angle OAB < \pi - \angle OAB - \angle BAA'' \equiv \angle A'AA'' \equiv \angle AOB'',$$

which means that B'' lies outside $\angle AOB$ and $A''A' \equiv B''A > BA$. As a result

$$B'A' \equiv B'A'' + A''A' \equiv BA + A''A' > 2 \cdot BA.$$

Expressed intuitively: *A point that moves with a constant velocity along ray $\overrightarrow{OA}$ distances itself at an accelerating rate from ray $\overrightarrow{OB}$.*

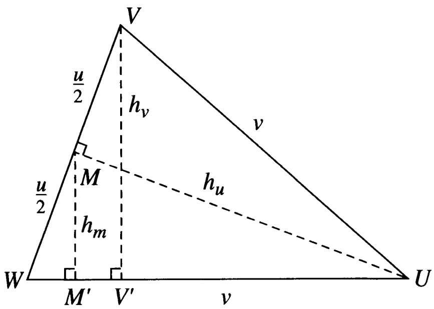

Figure 2

Consider now an acute-angled, isosceles triangle ΔUVW with apex U, having M as the midpoint of the base, and orthogonal projections M', V' of M, V on WU, as in FIGURE 2. The length of UM, the principal altitude of ΔUVW, will be denoted by h_u and the lengths of UW, VW, VV', and MM' by v, u, h_v, and h_m respectively. Now, if

Euclidean area measure were well-defined in the noneuclidean plane we would have

$$\text{in} \quad \triangle\, UVW, \qquad \frac{uh_u}{2} = \frac{vh_v}{2},$$

$$\text{and in} \quad \triangle\, UMW, \qquad \frac{(u/2)h_u}{2} = \frac{vh_m}{2},$$

and so $h_v = 2h_m$. However, we should have $h_v > 2h_m$, as we showed before in FIGURE 1, which means that at least one of the above two equations is false. Hence, Euclidean area measure is not applicable in noneuclidean geometry.

How does one prove that (base $\times$ altitude)$/2$ is well-defined in Euclidean geometry, and what accounts for the difference in noneuclidean geometry? As is often the case in Euclidean geometry, one verifies the equation between two products of segments by transforming it into one between two quotients and then applies a proportionality theorem [**3**, § 20]. And that is exactly what does not work in noneuclidean geometry; in FIGURE 1

$$\frac{B'A'}{OA'} > \frac{BA}{OA}.$$

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The Slope Mean and Its Invariance Properties

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For $a, b > 0$, we know that the arithmetic mean $A(a, b) = (a + b)/2$ produces the midpoint of the segment $[a, b]$ on the real line. But what if we interpret a and b as slopes? A more natural mean in this context could be the "intermediate" slope, specifically, the positive slope $S(a, b)$ of the line $y = S(a, b)x$ that bisects the angle formed by the lines $y = ax$ and $y = bx$. As $a, b > 0$ vary in the figure, one senses that $S(a, b)$ is different from $A(a, b)$, but nonetheless has characteristics often associated with a mean.