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On Tiling the n-Dimensional Cube

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Take a square and cut it into smaller pieces, each of which is a square. Now count the number of pieces. It is easy to show that the answer is not 2, 3, or 5, but it might be anything else. Asking the question a different way: for what values of k can one tile a square with k squares? Here *tiling* means covering with pieces that overlap only on edges. FIGURE 1 shows tilings of a square with $k = 4, 6, 7$, and 8.

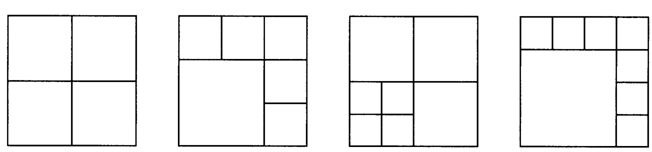

Figure 1 Some tilings of a square

It is an easy observation that if a tiling with k squares exists, then a tiling with $k + 3$ squares also exists. For in a tiling, a single square may be replaced by the first tiling in FIGURE 1. So, the existence of tilings for $k = 6, 7$, and 8, along with the easy passage from k to $k + 3$, provide an inductive proof of the existence of tilings for all $k \geq 6$. We leave as an exercise the impossibility of $k = 2, 3$, and 5. Hint: count corners.

Now we ask similar questions in higher dimensions. Can an n-dimensional cube (n-*cube* for short) be tiled with k smaller n-cubes?

FACT 1. The standard tiling of a unit n-dimensional cube with r^n cubes each of edge length $1/r$ shows that the answer is yes in dimension n for $k = 2^n, 3^n, 4^n, \ldots$.

Here is another key to solving the higher dimensional problem: Note that in a given k-tiling, a single tile may be replaced by a cube tiled with l tiles. This observation gives us Fact 2.

FACT 2. If the n-cube can be tiled with k cubes and can be tiled with l cubes, then it can be tiled with $k + l - 1$ cubes.

Now let $k = 1$ and $l = (2^n - 1)^n$ and apply Fact 2 j times to learn that the n-cube may be tiled with $1 + j[(2^n - 1)^n - 1] = j(2^n - 1)^n - j + 1$ cubes. As j varies from 0 to $2^n - 2$, the resulting numbers of tiles hit every congruence class modulo $2^n - 1$. For example, in dimension three, the situation is as in TABLE 1.

Now, by Fact 2, if a tiling exists with k tiles, then one exists for $k + 2^n - 1$ tiles, so if a tiling exists with k_1 tiles then a tiling exists with any larger $k_2 \equiv k_1 \bmod (2^n - 1)$. We conclude that tilings exist in dimension n for all k greater than or equal to

$$(2^n - 2)(2^n - 1)^n - (2^n - 2) + 1.$$

TABLE 1: Some possible tilings of the 3-cube

j	$j(2^3 - 1)^3 - j + 1 = 343j - j + 1$	mod 7
0	1	1
1	343	0
2	685	6
3	1027	5
4	1369	4
5	1711	3
6	2053	2

Now it is perhaps interesting to ask for the smallest number $f(n)$ so that an n-dimensional cube can be tiled with k n-dimensional cubes for all $k \geq f(n)$. We have shown $f(2) = 6$.

Examine TABLE 1. The values of k in the interval $[2047, 2053]$ are covered in the 7 congruence classes, but $k = 2046$ is not, since 2053 is the smallest number covered that is congruent to 2 mod 7. These 7 numbers, along with the k to $k + 7$ induction show that in fact $f(3) \leq 2047$.

Similarly, in n dimensions, the threshold is actually no larger than

$$(2^n - 2)(2^n - 1)^n - 2^n + 3 - (2^n - 2) = (2^n - 2)(2^n - 1)^n - 2^{n+1} + 5.$$

Our general upper bound for $f(n)$ is probably very bad. We will show that in fact $f(3) \leq 48$.

We must cover every congruence class modulo $7 = 2^3 - 1$. We accomplish this with the following values of k:

TABLE 2: Better ways to tile the 3-cube

k mod 7	0	1	2	3	4	5	6
k	49	1	51	38	39	54	20

Beginning with the simplest, we note that tiling with 1 cube is trivial. Tiling with 20 cubes can be done by taking a standard tiling of 27 congruent cubes and replacing the 8 cubes of a $2 \times 2 \times 2$ subcube with a single cube, as in FIGURE 2.

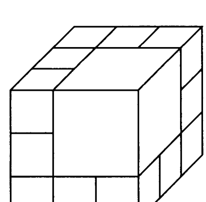

Figure 2 Using 20 cubes

Now we invoke Fact 2 with $k = l = 20$ to obtain a tiling with 39 cubes. This is shown in FIGURE 3. To construct a tiling with 38 cubes, begin with a standard tiling of 64 cubes and replace the 27 cubes of a $3 \times 3 \times 3$ subcube with a single cube.

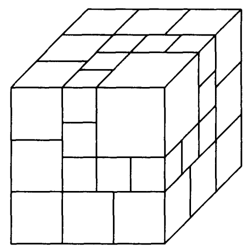

Figure 3 Tiling with 39 cubes

To construct a tiling with 49 cubes, imagine a unit cube sliced into three horizontal slabs of thickness $1/2$, $1/3$, and $1/6$. Tile the first slab with 4 cubes of side length $1/2$, the second with 9 cubes of side length $1/3$, and the third slab with 36 cubes of side length $1/6$. This is shown in FIGURE 4.

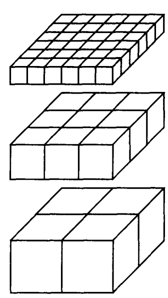

Figure 4 A 49er

Now tiling with 51 cubes is a bit trickier. Begin with a tiling of 20 cubes as above, recalling that 19 of the 20 are the same size. Look at the 6 faces of the cube and note that three are like the right-hand image in FIGURE 5 and three are like the left-hand one.

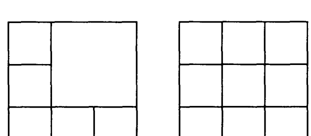

Figure 5 Faces of the 20 cube tiling

Pick two small cubes of the 20 cube tiling that share a face. Replace each of the two with a cube tiled with 20 in such a way that the shared face of each of the two looks like the right-hand picture in FIGURE 5. Now observe that 8 of the smallest cubes form a $2 \times 2 \times 2$ subcube, which can be replaced with a single cube. The result is a tiling with $20 + (20 - 1) + (20 - 1) - 8 + 1 = 51$ cubes.

The construction of a tiling with 54 cubes is similar. Recall our construction of a tiling with 38 cubes. Begin with a standard tiling using 8 cubes. Replace 2 cubes that share a common face with 38 cubes apiece so that the shared faces of the two are as

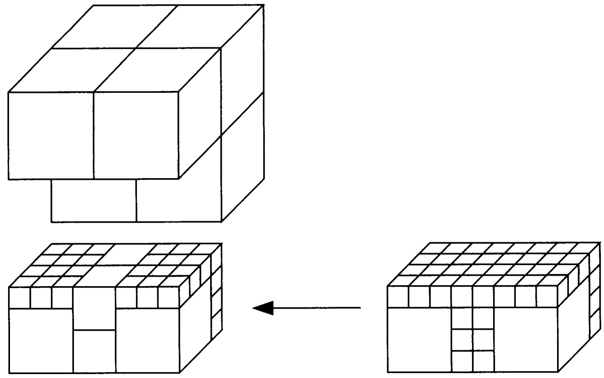

Figure 6 Tiling with 54 subcubes

in FIGURE 6. There are now four subcubes each consisting of 8 cubes. Each of these can be replaced by a single cube. This yields a tiling with $8 + (38 - 1) + (38 - 1) - 4(8 - 1) = 54$ cubes. We note that our constructions yield no tiling with 47 cubes, but that along with Fact 2, they yield tilings for all larger values of k.

Many interesting questions remain. What is the exact value of $f(n)$? Even $f(3)$ may be difficult. For $n = 2$ we know all values of k for which no tiling exists. For larger n one can easily see by counting corners that no tiling exists for $2 \leq k \leq 2^n - 1$. We conjecture that for $n \geq 3$ no tiling exists in dimension n for $2^n + 1 \leq k \leq 2^{n+1} - 2$. For $n = 3$, this conjecture can be proved with a lengthy examination of cases.

Acknowledgment. The referee has drawn our attention to an article by Hudelson [1], where most of our results have been anticipated, indeed with better bounds in four dimensions and higher.

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The Magic Mirror Property of the Cube Corner

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We show that a ray of light, successively reflected from three mutually perpendicular planes in ordinary three-space, comes back in the same direction it came from. This