and then

$$(\lambda_3, \mu_3, \nu_3) = (-\lambda_2, \mu_2, -\nu_2) = (\lambda, \mu, \nu).$$

The final result will be the same. This shows that the ray emerging after three reflections is parallel to the original ray.

Finally, we consider the special occurrence of a ray entering the cube corner parallel to one of its faces. This case can be viewed as if the reflections take place on a billiard table, with a ball hitting successively two concurrent sides of the table; the answer is the same.

Conclusion We have shown that (almost) every ray emerging from a cube corner system after reflecting off its three plane walls has the same direction as the incoming ray that produced it. Thanks to clever applications of this magic mirror property, our travel is safer and we can measure the distance all the way to the Moon. If you ever encounter Smart Dust, remember the reflective property behind its design.

REFERENCES

1. J. S. Allen, About bicycle reflectors contents, <http://www.bikexprt.com/bicycle/reflectors/>, last revised March 14, 2003, and all links, last revised April 27, 2002.
2. J. E. Faller and E. J. Wampler, The lunar laser reflector, *Scientific American*, **38**, March 1970, 38–50.
3. J. M. Kahn, R. H. Katz, and K. S. J. Pister, Emerging challenges: mobile networking for "Smart Dust," *J. Comm. and Networks* **2**:3 (2000), 188–196.
4. M. S. Scholl, Ray trace through a corner-cube retroflector with complex reflection coefficients, *J. Opt. Soc. Amer.* A **12** (1995), 1589–1592.
5. Chun-Ching Shih, Depolarization effect in a resonator with corner-cube reflectors, *J. Opt. Soc. Amer.* A **13** (1996), 1378–1384.
6. L. Zhou, J. M. Kahn, and K. S. J. Pister, Corner-cube retro-reflectors based on structure-assisted assembly for free-space optical communication, *IEEE J. Microelectromech. Syst.* **12**:3 (2003), 233–242.

Spherical Triangles of Area π and Isosceles Tetrahedra

JEFF BROOKS
JOHN STRANTZEN
La Trobe University
Victoria 3086, Australia
J.Brooks@latrobe.edu.au
J.Strantzen@latrobe.edu.au

There is a beautiful theorem in spherical geometry that is not well known, and that doesn't appear in most texts on the subject.

The theorem says that for any spherical triangle of area π on the unit sphere, four congruent copies of it tile the sphere; that is, the copies can be positioned so that any two triangles meet at an edge, with vertices meeting vertices, and the union of these four triangles is the sphere [**1**; **3**, p. 44; **6**, p. 94]. The configuration is shown in FIGURE 1, which may aid readers in guessing the proof.

What appears to be less known (in fact, the authors can find no reference to it) is that the four vertices on the sphere determined by this tiling form the vertices of an

isosceles tetrahedron (opposite edges of the tetrahedron are equal) [**2**]. Hence, every such tiling of the sphere is obtained as the projection from the center of an inscribed isosceles tetrahedron.

Background We introduce some preliminary definitions and results that the reader may already know, in which case it is possible to skip ahead. Given a point P on the unit sphere, the intersection of the line joining the center of the sphere to P meets the sphere at a diametrically opposite point P'. The pair P and P' are called *antipodal points*. We define a *line* on the sphere to be a great circle, that is, the intersection of the sphere with a plane containing the center of the sphere. The complement of a line is two disjoint open hemispheres.

Given two distinct nonantipodal points, P and Q, there is a unique line containing them. This line is the intersection of the sphere with the plane containing P and Q and the center of the sphere. We denote it by lPQ. We let $\overline{PQ}$ denote the smaller of the arcs on the line that connects P to Q. It follows that length of $\overline{PQ}$ is less than π. We denote the fact that two arcs $\overline{PQ}$ and $\overline{RS}$ have equal length by $\overline{PQ} = \overline{RS}$, blurring the distinction between a segment and its measurement.

Given three distinct noncollinear points A, B, C, no two of which are antipodal, we define the *spherical triangle* $\triangle ABC$ to be the union of the arcs $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. We call $\overline{AB}$ the *edge* of the triangle joining vertex A to vertex B, similarly for $\overline{BC}$ and $\overline{CA}$. Such a triangle necessarily lies in an open hemisphere. Clearly each edge of $\triangle ABC$ determines an open hemisphere containing the remaining vertex. The intersection of these three open hemispheres is the interior of $\triangle ABC$. The area of $\triangle ABC$ is defined to be the area of its interior. It should be intuitive as to what is meant by the three angles of $\triangle ABC$, and we denote by $\angle CAB$, $\angle ABC$, and $\angle BCA$ the measure in radians of these three angles. The main result that we use to prove our theorem is a remarkable theorem attributed to Girard. Girard's theorem says that

$$\angle CAB + \angle ABC + \angle BCA - \pi = \text{Area}(\triangle ABC)$$

Weeks [**8**] gives an elegant proof of this result.

Finally, we need to say what we mean by two triangles being congruent. We adopt the Euclidean notion, which says that two triangles are congruent if there is an isometry between the triangles. The isometries of the sphere are of three types, a reflection in a line, a product of two reflections (this is equivalent to a rotation about an axis through the antipodal points where the lines meet), and a reflection followed by a rotation perpendicular to the line of reflection (a glide reflection) [**5**].

Reflections and glide reflections do not preserve orientation, which we wish to do in this discussion. Hence, for us two triangles are congruent if one can be rotated onto the other, see [**7**]. The usual theorems for congruence of spherical triangles—SSS, SAS, ASA, and AAA—will determine uniqueness up to isometry, so we must show that the copies of our triangle are all rotations of it.

Proving the theorem To demonstrate our result, let $\triangle ABC$ be a spherical triangle of area π. Choose an edge (say $\overline{AB}$), let M be the midpoint of that edge, as in FIGURE 1.

From our definition of a triangle it follows that $\overline{MC}\setminus\{M, C\}$ lies in the interior of $\triangle ABC$. Rotate $\triangle ABC$ by π about the axis through M and its antipodal point M'. Under this rotation A and B interchange and C maps to a fourth distinct point D. It is clear that C and D are distinct, since the length of $\overline{MC}$ is less than π. Also C, M, and D lie on the line determined by M, M', and C.

It is not difficult to see that $\overline{MC}$ has a length greater than $\pi/2$ otherwise $\triangle ABC$ would fit inside a quadrant (as the length of $\overline{AB}$ is less than π) and hence have area

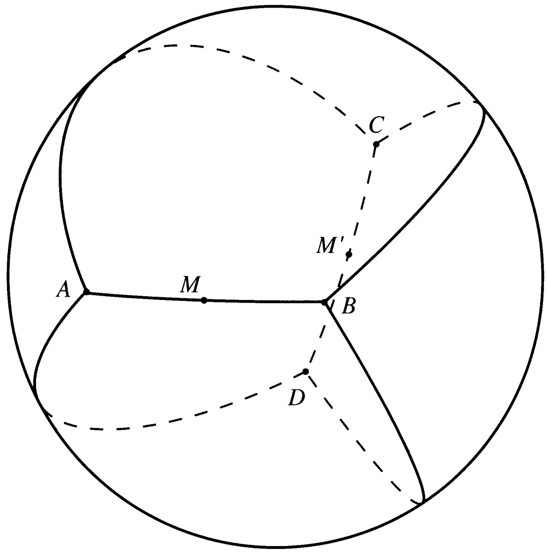

Figure 1

strictly less than π. Hence, M' is the midpoint of $\overline{CD}$. The points C, M, and D lie on the line determined by M, M', and C and $\overline{M'D} = \overline{M'C}$. Also M, M', and A lie on line lAB, with A and B lying in distinct open hemispheres determined by line lCD.

Consider the four triangles $\triangle ABC$, $\triangle BAD$, $\triangle CDA$, and $\triangle DCB$. Clearly any two meet at a common edge with vertices meeting vertices. If we partition $\triangle CDA$ into $\triangle CM'A$ and $\triangle AM'D$, and $\triangle DCB$ into $\triangle DM'B$ and $\triangle BM'C$, we see that the three triangles $\triangle CM'A$, $\triangle ABC$, and $\triangle BM'C$ tile the closed hemisphere determined by line AB and C. Similarly $\triangle AM'D$, $\triangle BAD$, and $\triangle DM'B$ tile the closed hemisphere determined by line lAB and D. It follows that $\triangle ABC$, $\triangle BAD$, $\triangle CDA$, and $\triangle DCB$ tile the sphere.

We have $\triangle ABC$ congruent to $\triangle BAD$, as they are rotations of each other by π about the axis through M, M'. Also $\triangle CDA$ is congruent to $\triangle DCB$ for the same reason. By Girard's theorem, we have $\angle CAB + \angle ABC + \angle BCA = 2\pi$ (since Area $\triangle ABC = \pi$). Also, $\angle CAB + \angle DAC + \angle BAD = 2\pi$ (sum of angles around vertex A). But $\angle BAD = \angle ABC$ ($\triangle ABC$ is congruent to $\triangle BAD$) hence $\angle DAC = \angle BCA$.

It now follows that if we rotate $\triangle ABC$ by π about the midpoint N of $\overline{AC}$, then A and C change places and B maps to D ($\overline{DA} = \overline{CB}$ as $\triangle CDA$ is congruent to $\triangle DCB$). Thus $\triangle ABC$ is congruent to $\triangle CDA$ and so the four triangles are congruent. In particular $\overline{AB} = \overline{CD}$, $\overline{BC} = \overline{AD}$ and $\overline{AC} = \overline{BD}$, which means that the tetrahedron formed by the vertices $ABCD$ is isosceles.

Now the great circle through M, M', and the midpoint N of AC has $\overline{MN} = \overline{M'N}$ ($\triangle ABC$ is congruent to $\triangle CDA$), and as $\overline{MN} + \overline{M'N} = \pi$, it follows that $\overline{MN} = \pi/2$. We therefore deduce that the medial triangle of $\triangle ABC$ is an octant [**1, 4**].

Constructions This proof gives a method for drawing such triangles on a sphere: Mark two antipodal points on the sphere. Center congruent arcs of length less than π on these points in such a way that the arcs do not lie on the same great circle. Connect the end points of the arcs in the appropriate way to obtain the tiling. Varying the length of the arcs and the angle they make with each other gives all possible tilings.

To construct an isosceles tetrahedron, note that its faces must be four congruent acute angled triangles. Take any acute angled triangle ABC and rotate it by π about an axis through the midpoint of $\overline{AB}$ perpendicular to the plane of the triangle. Let D be

the image of C. Then $\overline{AB} < \overline{CD}$. Now rotate $\triangle ABD$ about $\overline{AB}$ so that the image of D is D' and $\overline{AB} = \overline{CD'}$. It follows that tetrahedron $ABCD'$ is isosceles.

Acknowledgment. The authors would like to thank Paul Pontikis for the typing, Jeanette Varrenti and Marcel Jackson for the electronic picture.

REFERENCES

1. Grant Cairns, Margaret McIntyre, and John Strantzen, Geometric proofs of some recent results of Yang Lu, this MAGAZINE **66**:4 (October 1993), 263–265.
2. N. A. Court, *Modern Pure Solid Geometry*, Chelsea Publishing Co., New York, 1964.
3. H. L. Davies, Packings of spherical triangles, *Proceedings of The Colloquium on Convexity*, Copenhagen, August 1965, pp. 42–51.
4. W. J. M'Clelland and T. Preston, *A Treatise on Spherical Trigonometry with Applications to Spherical Geometry Part I*, Macmillan Publishing Co., New York, 1907, *and Part II*, 1909.
5. Patrick J. Ryan, *Euclidean and Non-Euclidean Geometry, An Analytic Approach*, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1986.
6. D. M. Y. Sommerville, Division of space by congruent triangles and tetrahedra, *Proceedings of The Royal Society of Edinburgh*, **43** (1923) pp. 85–116.
7. I. Todhunter and J. G Leatham, *Spherical Trigonometry*, Macmillan and Co. Limited, St. Martin's Street, London, 1943.
8. Jeffrey R. Weeks, *The Shape of Space*, 2nd ed., Marcel Decker, Inc., New York, Basel, 2002.

A Butterfly Theorem for Quadrilaterals

SIDNEY KUNG
University of North Florida
Jacksonville, FL 32224
sidneykung@yahoo.com

One of the more appealing theorems in plane geometry is the butterfly theorem:

BUTTERFLY THEOREM. *Through the midpoint I of a chord AC of a circle, two other chords EF and HG are drawn. If EG and HF intersect AC at M and N, respectively, then $IM = IN$.*

Since 1815, when this theorem appeared as a proposed problem in the *Gentleman's Diary* (1815, pp. 39–40; see also [**1**, p. 195]) it has attracted many lovers of mathematics, some of whom have produced simple and elegant proofs, while others devised various generalizations. In a delightful and well-documented article, Bankoff [**1**] discusses the butterfly theorem for circles and some variants. In particular, on p. 207 one finds an "area method" applied to prove that if I is a point anywhere on the chord AC (as in FIGURE 1), $IA = a$, $IC = c$, $IM = m$, $IN = n$, then:

$$\frac{1}{m} - \frac{1}{n} = \frac{1}{a} - \frac{1}{c} \tag{1}$$

In this note, we give a similar proof of a butterfly theorem for quadrilaterals. Our proof depends primarily upon the following properties for areas of triangles:

P1 If K is the intersection of the lines XY and UV, $V \neq K$ (FIGURE 2a), then

$$\frac{\mathcal{A}(UXY)}{\mathcal{A}(VXY)} = \frac{UK}{VK}, \quad \text{where } \mathcal{A}(UXY) \text{ denotes the area of triangle } UXY.$$