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The Circumdisk and its Relation to a Theorem of Kirszbraun and Valentine

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In this note we consider the following problem: *Given a finite set of points in euclidean m-space, characterize the radius R of the smallest disk (closed solid sphere) which contains those points.* We believe that our solution to this problem is new, characterizing R in terms of a well-known quadratic form. Moreover, it provides a new proof of the very appealing theorem of Kirszbraun-Valentine: *If a collection of disks (of varying radii) in E^m having nonempty intersection are rearranged so that corresponding distances between centers do not increase, then the rearranged collection also has nonempty intersection.* Whether or not the volume of the intersection can decrease remains a problem which baffles mathematicians.

We first need to review some of the basic ideas that will be required. Let $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$, $n \geq 1$, be a collection of distinct points in euclidean space E^m. Among all those disks (closed solid spheres) which contain these points, there is a unique disk of smallest radius, called the **circumdisk**. In Figure 1 we give a simple but very useful example. Here $m = 2$, $n = 2$, and the unbroken circle with diameter $|\mathbf{p}_1 - \mathbf{p}_2|$ bounds the circumdisk.

The existence of a minimal containing disk follows from the Blaschke selection theorem [**12**, p. 37]. However, those readers familiar with sequential compactness in E^m can easily concoct a proof of existence. Such a disk is unique, for if there were two distinct minimal disks, centered at $\mathbf{u}$ and $\mathbf{u}'$, respectively, and having radius R, then a disk of radius $\sqrt{R^2 - \frac{1}{4}|\mathbf{u} - \mathbf{u}'|^2}$ centered at $\frac{1}{2}(\mathbf{u} + \mathbf{u}')$ would also contain all the points $\mathbf{p}_i$.

Some writers use the word circumsphere instead of circumdisk. However, we wish to reserve the former as a term for a generalization of the notion **circumcircle**. Thus, if $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$ are points in E^m, we define the **circumsphere** to be the sphere of least radius on which all the points lie, provided such a sphere exists. For example, three collinear points have no circumsphere. In Figure 1, observe that $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ lie on a circumsphere (the circle determined by the dashed arc)

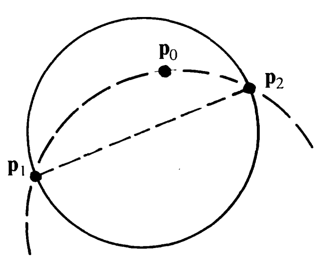

Figure 1

which is not the boundary of the circumdisk. If a circumsphere exists, its uniqueness is established in the same manner as for the circumdisk.

Recall that a set is **convex** if for any two points in the set, the line segment joining them also lies in the set. While our claims concerning convexity should be clear at least for the plane ($m=2$), the excellent references [2] and [12] may be consulted for the general theory of convex sets in E^m. The **convex hull** of the points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ is the intersection of all convex sets containing them. It is easily checked that the convex hull consists of all points $\mathbf{p} = x_0\mathbf{p}_0 + \cdots + x_n\mathbf{p}_n$, where the x_i are nonnegative real numbers and $x_0 + x_1 + \cdots + x_n = 1$. It is clear that the circumdisk contains the convex hull, and that at least two of the points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ lie on the boundary of the circumdisk (FIGURE 1 illustrates this). Those points which lie on this boundary will be called ***c*-extreme points**.

Our first theorem is standard, but a discussion is included for completeness and for future reference. Lemma 1, which actually gives a metric characterization of the convex hull, has various applications in geometric extremal problems. It says that if the point $\mathbf{u}$ lies in the convex hull of a finite set of points $\{\mathbf{q}_i\}$, then given any other point $\mathbf{u}'$, some one of the $\mathbf{q}_i$ is closer to $\mathbf{u}$ than to $\mathbf{u}'$.

LEMMA 1. *Let the point* $\mathbf{u}$ *lie in the convex hull of the points* $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_s$. *If* $\mathbf{u}'$ *is distinct from* $\mathbf{u}$, *then for some* i, $|\mathbf{u} - \mathbf{q}_i| < |\mathbf{u}' - \mathbf{q}_i|$.

To prove this, choose H to be the $(m-1)$ flat (or hyperplane) through $\mathbf{u}$ which is perpendicular to the segment $\overline{\mathbf{u}\mathbf{u}'}$. Then for at least one value of $i, \mathbf{q}_i$ must lie in the closed halfspace of H which does not contain $\mathbf{u}'$; this choice of i works.

THEOREM 1. *Let* $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ *be points in* E^m. *Then*

(a) *the center of the circumdisk is contained in the convex hull of the c-extreme points*,
(b) *the circumdisk of the c-extreme points is the circumdisk of the set of points* $\mathbf{p}_i$, *and*
(c) *the boundary of this disk is the circumsphere of the c-extreme points.*

All of the statements in Theorem 1 are illustrated in FIGURE 1.

Proof. (a). Let $\mathbf{u}$ be the center of the circumdisk which has radius R, and suppose that $\mathbf{u}$ does not lie in the convex hull K of the c-extreme points. Let $\delta_1 > 0$ be the distance from $\mathbf{u}$ to K, and let $\delta_2 > 0$ be the least of all distances from $\mathbf{u}$ to those $\mathbf{p}_i$ which are not c-extreme points. Since K is convex and closed, there will be exactly one point $\mathbf{u}_1$ in K such that $|\mathbf{u} - \mathbf{u}_1| = \delta_1$. Choose the point $\mathbf{u}_2$ on the segment $\overline{\mathbf{u}\mathbf{u}_1}$ so that $|\mathbf{u} - \mathbf{u}_2| = \delta$, where $\delta = \frac{1}{2}\min(\delta_1, \delta_2)$. A contradiction is obtained by observing that all points $\mathbf{p}_i$ are contained in a disk centered at $\mathbf{u}_2$ of radius $\sqrt{R^2 - \delta^2}$.

(b) and (c). Lemma 1, applied to the center $\mathbf{u}$, may now be used to deduce that no disk, centered at $\mathbf{u}'$, of smaller radius than R can contain the c-extreme points. Lemma 1 also shows that the c-extreme points can lie on no smaller sphere than the boundary of the circumdisk.

Theorem 1 can be made sharper; a theorem of Carathéodory can be invoked to show that the center of the circumdisk is contained in the convex hull of $m+1$ or fewer c-extreme points. This strengthened version is not needed for the present work, but a discussion of Carathéodory's theorem may be found in Eggleston's book [2].

We next introduce the quadratic form

$$Q = \sum_{i,j=0}^{n} |\mathbf{p}_i - \mathbf{p}_j|^2 x_i x_j. \tag{1}$$

For many purposes it is convenient to rewrite Q as

$$Q = \sum_{i,j=0}^{n} \left(|\mathbf{p}_i|^2 + |\mathbf{p}_j|^2\right) x_i x_j - 2\left|\sum_{i=0}^{n} x_i \mathbf{p}_i\right|^2. \tag{2}$$

The transition from (1) to (2) is accomplished by using the identity

$$|\mathbf{p}_i - \mathbf{p}_j|^2 = |\mathbf{p}_i|^2 + |\mathbf{p}_j|^2 - 2\mathbf{p}_i \cdot \mathbf{p}_j$$

and the fact that the dot product is linear. It is clear from (1) that the value of Q at $(x_0, x_1, \ldots, x_n)$ is independent of the choice of coordinate system in E^m.

LEMMA 2. *Suppose* $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ *lie in a disk of radius* R, *and the nonnegative numbers* x_i *satisfy* $x_0 + x_1 + \cdots + x_n = 1$. *Then the quadratic form* Q *in* (1) *satisfies*

$$\operatorname{Max} Q \leqslant 2R^2. \tag{3}$$

To prove this, choose the center of the disk as the origin and note from (2) that

$$Q \leqslant \sum \left(|\mathbf{p}_i|^2 + |\mathbf{p}_j|^2 \right) x_i x_j \leqslant 2R^2 \left(\sum x_i \right)^2 = 2R^2.$$

LEMMA 3. *Suppose the points* $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_s$ *possess a circumsphere of radius* R. *Then subject to* $x_0 + x_1 + \cdots + x_s = 1$, *the quadratic form* Q *in* (1) *satisfies*

$$\operatorname{Max} Q = 2R^2.$$

Moreover, a nonnegative vector ($x_i \geqslant 0$ *for each* i) *which maximizes* Q *exists when the center of the circumsphere lies in the convex hull of the* $\mathbf{q}_i$.

Proof. If the center of the circumsphere is the origin, then (2) implies

$$Q = 2R^2 - 2\left| \sum_i x_i \mathbf{q}_i \right|^2. \tag{4}$$

The center of the circumsphere must lie in the flat spanned by the $\mathbf{q}_i$, or else a reflection through this flat would contradict the uniqueness of the circumsphere. Thus, there is a suitable vector $(x_0, x_1, \ldots, x_s)$ such that $\Sigma x_i \mathbf{q}_i = 0$. If the center lies in the convex hull of the $\mathbf{q}_i$, then the x_i may be chosen nonnegative.

Using these results, we can now easily characterize the radius of the circumdisk of a pointset in terms of Q.

THEOREM 2. *Suppose that the points* $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ *in* E^m *possess a circumdisk of radius* R. *Then subject to* $x_i \geqslant 0$ *for each* i, *and* $x_0 + x_1 + \cdots + x_n = 1$, *the quadratic form* Q *in* (1) *satisfies*

$$\operatorname{Max} Q = 2R^2.$$

Lemma 2 says that Max $Q \leqslant 2R^2$. However, if we set $x_i = 0$ whenever $\mathbf{p}_i$ is not c-extreme, then Theorem 1 and Lemma 3, applied to the c-extreme points, imply that there actually is a suitable vector $(x_0, x_1, \ldots, x_n)$ at which $Q = 2R^2$.

COROLLARY. *Let* $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ *and* $\mathbf{p}'_0, \mathbf{p}'_1, \ldots, \mathbf{p}'_n$ *be points in* E^m *such that*

$$|\mathbf{p}'_i - \mathbf{p}'_j| \leqslant |\mathbf{p}_i - \mathbf{p}_j| \text{ for all } i, j. \tag{5}$$

Then the circumdisk of the $\mathbf{p}'_i$ is no larger than that of the $\mathbf{p}_i$.

Proof. Let $y_0, y_1, \ldots, y_n$, $y_i \geqslant 0$, $y_0 + y_1 + \cdots + y_n = 1$, be chosen to maximize the form Q' associated with the points $\mathbf{p}'_i$. Then

$$2R'^2 = Q'(y_0, y_1, \ldots, y_n) \leqslant Q(y_0, y_1, \ldots, y_n) \leqslant 2R^2.$$

It may be seen that, subject to (5), $R = R'$ if and only if there is a subset of the c-extreme points among the $\mathbf{p}_i$, containing the center of the circumdisk in its convex hull, which is congruent to a corresponding subset of the $\mathbf{p}'_i$.

There are several well-known results due to Kirszbraun **[5]** and Valentine **[11]** on intersections of disks. The following theorem (or its equivalent), described in our introduction, is a consequence of the Corollary.

THEOREM 3. *Suppose the disks $D_0, D_1, \ldots, D_n$ in E^m with respective radii $R_0, R_1, \ldots, R_n$ and centers $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ have nonempty intersection. If the disks are rearranged to be centered at $\mathbf{p}'_0, \mathbf{p}'_1, \ldots, \mathbf{p}'_n$ in such a manner that $|\mathbf{p}'_i - \mathbf{p}'_j| \leqslant |\mathbf{p}_i - \mathbf{p}_j|$ for all i, j, then the intersection of the rearranged disks remains nonempty.*

Proof. If all the disks have the same radius R, the theorem follows immediately from the Corollary, since a collection of disks of radius R has nonempty intersection if and only if the circumdisk of their centers has radius not exceeding R. (Incidentally, this shows that the Corollary itself follows at once from Theorem 3.) We now utilize a simple "Chinese checkers" geometric construction to deduce the theorem from this special case.

Choose R greater than the maximum of the R_i. Place $2n$ disks of radius R in E^{m+1}, centered at the $2n$ points $(\mathbf{p}_i, \pm\sqrt{R^2 - R_i^2})$. Note that the intersection of these disks of radius R is a convex set which is symmetric with respect to the hyperplane E^m (or $x_{m+1} = 0$), and which is nonempty since its intersection with E^m is the intersection of the original disks in E^m of varying radii. (See FIGURE 2.) Likewise place $2n$ disks of radius R centered at the $2n$ points $(\mathbf{p}'_i, \pm\sqrt{R^2 - R_i^2})$. Observe that the two sets of centers of disks of radius R satisfy the distance inequality, thus the rearranged disks have nonempty intersection. Since this intersection is a convex set which is symmetric with respect to the hyperplane E^m, it follows that the disks in E^m centered at the $\mathbf{p}'_i$ also have nonempty intersection.

Some versions of Theorem 3 state (at least implicitly) that if a collection of disks has nonempty intersection, then the intersection contains a point in the convex hull of the centers. Theorem 1 guarantees this to be true when all radii are equal. The geometric construction described above quickly extends the validity to the situation of varying radii.

A number of proofs and generalizations of the theorems of Kirszbraun and Valentine have appeared; of these, the note of Schoenberg **[10]** is probably the most closely related to the present work. The articles of Mickle **[7]** and Grünbaum **[3]**, **[4]** should be consulted for elegant extensions of the Kirszbraun-Valentine theory. A very pretty generalization of spheres having nonempty intersection, due to G. Minty, is also described in **[4]**.

The relation of the form Q to the theory of metric geometry was investigated by Schoenberg in a series of very interesting papers; **[9]** provided an early elegant example. More recently the author **[1]** has been attempting to find practical methods of employing the form Q in the solution of geometric problems in E^m. Problem 7 in the article by Klee **[6]** in this *Magazine* provided impetus for the present note.

We close with some questions. Suppose $D_0, D_1, \ldots, D_n$ are unit disks in the plane. If the centers are moved closer together, can the perimeter of the intersection decrease? Consideration of the evolute of the boundary of the intersection leads to further problems of interest. Are there generalizations of Helly's theorem which might be helpful? The theorem of Sallee **[8]** could be a step in the right direction.

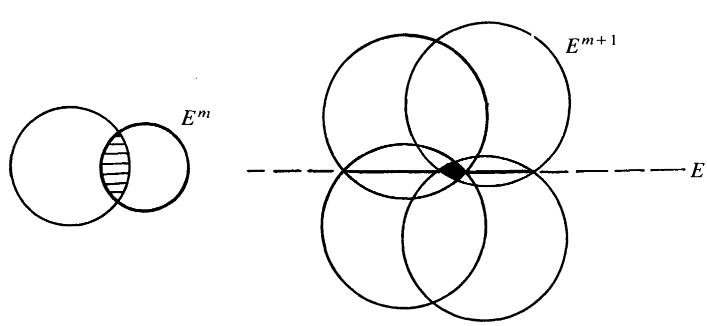

FIGURE 2

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Is Every Continuous Function Uniformly Continuous?

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In any elementary treatment of uniform continuity, examples are given to show that the answer to our title question is *no*. The most often cited functions are probably

$$f: (0,1] \to R$$
$$x \to 1/x$$

and

$$g: R \to R$$
$$x \to x^2.$$

To show that these functions are continuous but not uniformly continuous it is assumed that the metric is the usual Euclidean or standard one, i.e., the distance between two real numbers x and y is $d_s(x, y) = |x - y|$. In this note we are concerned with finding different metrics on R so that not only are continuous functions (like the above) also uniformly continuous, but so that the new metrics are *equivalent* to d_s.

First, we recall the definitions of continuity and uniform continuity. Let (X, d_1) and (Y, d_2) be metric spaces (see **[10]**). A function $f: D \to Y$, where $D \subseteq X$, is (d_1, d_2)-**continuous at a point** a in D if: for each $\varepsilon > 0$, there exists a $\delta_a > 0$ such that

$$x \in D \quad \text{and} \quad d_1(x, a) < \delta_a \text{ imply } d_2(f(x), f(a)) < \varepsilon.$$

The function f is (d_1, d_2)-**continuous** if it is (d_1, d_2)-continuous at every point of its domain D. In contrast, a function $f: D \to Y$ is (d_1, d_2)-**uniformly continuous** if: for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x, y \in D \quad \text{and} \quad d_1(x, y) < \delta \text{ imply } d_2(f(x), f(y)) < \varepsilon.$$