Let n be a nonnegative integer. Then on $(0, 1]$, $\sqrt{\frac{x^2}{x^2+1}}$ and x^n are increasing functions, hence their composition is increasing and we have

$$\begin{aligned}
&\sup\{|f^{-(n)}(x)g^{(n)}(x)| : 0 < x < 1\} \\
&\quad = \sup\left\{\left|\frac{x^n}{n!}\frac{n!}{(x+a)^{n+1}}\right| : 0 < x < 1\right\} \\
&\quad = \sup\left\{\left(\sqrt{\frac{x^2}{x^2+1}}\right)^n \cdot \frac{1}{\sqrt{x^2+1}} : 0 < x < 1\right\} \\
&\quad \le \left(\frac{1}{\sqrt{2}}\right)^n \cdot 1 \to 0
\end{aligned}$$

Thus by (5) above we see this i.p. series indeed converges to π.

Since the convergence criteria for an i.p. series are so easily satisfied and because of the variety in their form, i.p. series have the potential for wide application. Due to the elementary nature of integration by parts and infinite series they offer new topics for the classroom and projects for advanced students. They give the working mathematician an elegant method for deriving series and the potential for discovering new ones.

What Fraction of a Soccer Ball Is Covered with Pentagons?

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The surface of many soccer balls is covered with pentagons and hexagons in such a way that one pentagon and two hexagons meet at each vertex, there being twelve pentagons and twenty hexagons altogether (see Fig. 1). The pentagons are generally set off in a different color to give the ball some contrast and make it easier to see. While watching the World Cup soccer matches last summer, I found myself wondering what fraction of the surface of a soccer ball is covered with pentagons. It is the purpose of this note to give the answer to this geometrical puzzle.

A rough answer to the puzzle can be obtained using Euclidean geometry if one assumes that the pentagons and hexagons on the ball are all planar. If l denotes the common edge length of the (planar) pentagons and hexagons, the area of a pentagon is $A_p = (5l^2/4)\cot(\pi/5)$ and that of a hexagon $A_h = 3\sqrt{3}l^2/2$. Letting $\phi = 2\cos(\pi/5) = \left(1+\sqrt{5}\right)/2$ be the golden ratio, the fraction of the ball covered with pentagons can be worked out as

$$F = \frac{12A_p}{12A_p + 20A_h} = \frac{\phi}{\phi + \sqrt{48 - 12\phi^2}} \cong 0.28435. \tag{1}$$

Figure 1 A common type of soccer ball, covered with 12 pentagons and 20 hexagons.

However the pentagons and hexagons are not planar, and so the accuracy of this estimate is open to question. It would be nice to have an exact answer to compare to the above result.

One way of getting the exact answer is to use a theorem of spherical geometry, according to which the area of a spherical triangle is equal to the product of its "angular excess" (i.e. the amount by which the sum of its angles, in radians, exceeds π) and the square of the radius of the sphere on which it lies [**1**]. This theorem can be used to calculate the area of a pentagon on a soccer ball as five times the area of one of the elemental triangles into which it is divided by the great circle arcs that join its center to its vertices. Let us take the radius of the soccer ball to be unity and denote by θ_p the vertex angle of a pentagon on it. Then the angles of the elemental triangle of which the pentagon is made up are $2\pi/5$, $\theta_p/2$, and $\theta_p/2$, and the area of this triangle is $2\pi/5 + \theta_p/2 + \theta_p/2 - \pi = \theta_p - 3\pi/5$, from which it follows that the area of the pentagon is $5(\theta_p - 3\pi/5)$. The fractional area occupied by the pentagons is therefore

$$F = \frac{12 \cdot 5 \cdot (\theta_p - 3\pi/5)}{4\pi} = \frac{15}{\pi}\left(\theta_p - \frac{3\pi}{5}\right). \tag{2}$$

The above formula suggests an empirical method of determining F, based on measuring the angle θ_p on a soccer ball. However this method proves to be unsatisfactory because $(\theta_p - 3\pi/5)$, the difference in the vertex angles of the spherical and planar pentagons, is only on the order of a few degrees and requires θ_p to be measured to a small fraction of a degree if F is to be calculated accurately via Eq. (2). Needless to say, most soccer balls are not put together with this end in mind!

An alternative approach to calculating F is based on a formula for the area of a spherical triangle due to Euler and Lagrange. Let $\overline{a}$, $\overline{b}$, and $\overline{c}$ be vectors from the center of a unit sphere to the vertices of a spherical triangle on it. We assume that the triangle is an *Euler triangle*, i.e., one in which no side or angle (both expressed in radian measure) exceeds π. Then the area, Ω, of this triangle is given by

$$\tan\left(\frac{\Omega}{2}\right) = \frac{|\overline{a} \cdot \overline{b} \times \overline{c}|}{1 + \overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{c} + \overline{c} \cdot \overline{a}}. \tag{3}$$

A derivation of this formula, together with some of its history, can be found in Erikkson [**2**]. The reader can convince him(her)-self of the correctness of this formula in at least one special case by applying it to a spherical triangle all of whose angles are right angles (and for which the unit vectors $\overline{a}$, $\overline{b}$, and $\overline{c}$ are mutually orthogonal), for which it yields the expected result $\Omega = \pi/2$. Let (θ, ϕ) be the (polar, azimuthal) angles of a point on the surface of the soccer ball, and take the angular coordinates of the center of a pentagonal face and two of its adjacent vertices to be $(0, 0)$, $(\theta_1, 0)$, and $(\theta_1, 2\pi/5)$, respectively. The (unit) vectors from the center of the soccer ball to these vertices are then $\overline{a} = (0, 0, 1)$, $\overline{b} = (\sin\theta_1, 0, \cos\theta_1)$, and $\overline{c} = [\sin\theta_1 \cos(2\pi/5), \sin\theta_1 \sin(2\pi/5), \cos\theta_1]$. Substituting these into (3) allows one to calculate the area of an elemental triangle, from which one can get F. All that is needed to carry out this calculation is knowledge of the angle θ_1.

It is at this point that one needs to delve a little more deeply into the geometry of a soccer ball. A soccer ball is modeled on a truncated icosahedron, obtained by slicing off the corners of a regular icosahedron in such a way that each of its twelve vertices gets replaced by a regular pentagon and each of its twenty (triangular) faces by a regular hexagon. For this to happen, it is necessary that only the central third of each edge of the icosahedron be retained (as one of the edges of a resulting hexagonal face), with the thirds at either end being discarded to make way for the new pentagonal faces. If the edges of the truncated icosahedron are then projected on to its circumscribing sphere in such a way that each edge goes into a great circle arc on the sphere, the soccer ball pattern results. The reader who wishes to study the geometry of a truncated icosahedron in more detail can consult [**3**], which also gives instructions for building one. Wenninger [**4**] has a nice diagram showing how projecting a truncated icosahedron on to its circumscribing sphere leads to a pattern similar to that seen on the surface of a soccer ball.

It is known [**1**, Chapter 10] from the geometry of a regular icosahedron that the angle subtended by one of its edges at its center is $\theta_0 = \arctan(2) \cong 63.43°$. Taking two adjacent vertices of an icosahedron inscribed in a unit sphere to have coordinates $\overline{u}_1 = (0, 0, 1)$ and $\overline{u}_2 = (\sin\theta_0, 0, \cos\theta_0)$, one finds that the two vertices of the soccer ball lying on the joining edge are $\overline{v}_1 = \frac{2}{3}\overline{u}_1 + \frac{1}{3}\overline{u}_2$ and $\overline{v}_2 = \frac{1}{3}\overline{u}_1 + \frac{2}{3}\overline{u}_2$. The angle θ_1 can then be calculated as the angle between the vectors $\overline{u}_1$ and $\overline{v}_1$, and the common edge length of a pentagon or hexagon on the soccer ball, which we will denote θ_s, as the angle between the vectors $\overline{v}_1$ and $\overline{v}_2$. A simple calculation involving dot products shows that

$$\theta_1 = \arccos\left[\sqrt{\frac{8\phi + 17}{8\phi + 21}}\right] \cong 20.08° \quad \text{and} \quad \theta_s = \arccos\left(\frac{8\phi + 1}{10\phi - 1}\right) \cong 23.28°, \tag{4}$$

where we repeatedly used the relation $\phi^2 = \phi + 1$ to cast the above expressions in the simplest form possible. It is evident from this construction that $2\theta_1 + \theta_s = \theta_0$ (as is also evident numerically). With θ_1 in hand, we can calculate F from (3) in the manner indicated earlier and find that

$$F = \frac{30}{\pi} \arctan\left[\frac{\sin^2\theta_1 \sin(2\pi/5)}{(1 + \cos\theta_1)^2 + \sin^2\theta_1 \cos(2\pi/5)}\right] \cong 0.28177. \tag{5}$$

An alternative way of calculating F is to calculate the vertex angle θ_p of a pentagon on the soccer ball and then use it in (2). One can find θ_p from θ_1 and θ_s by using the cosine rule of spherical trigonometry, according to which the angle A opposite the side a of a spherical triangle with sides a, b, and c (in radian measure) is given by

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{6}$$

Applying this to an elemental triangle within a pentagonal face, with $A = \theta_p/2$, $a = b = \theta_1$, and $c = \theta_s$, gives

$$\theta_p = 2\arccos\left[\frac{\cos\theta_1 - \cos\theta_1\cos\theta_s}{\sin\theta_1\sin\theta_s}\right] \cong 111.38^\circ. \tag{7}$$

Using the radian measure of this angle in (2) gives $F = 0.28177$ to five decimal places, which agrees with (5) and gives us additional confidence in this result.

A comparison of (2) and (5) shows that the "planar approximation" used in getting (2) is remarkably good and gives a value just about 1 percent higher than the true value. The makers of soccer balls are evidently well aware of this close convergence, because they put the ball together out of planar pentagonal and hexagonal patches. After the patches are sewn together and the ball is inflated, the patches flex gently to accommodate themselves to the demands of spherical geometry.

Despite the near equality of (2) and (5), it is worth noting that the vertex angles of the spherical pentagons and hexagons on a soccer ball differ appreciably from those of the planar pentagon and hexagon. The vertex angles of the spherical pentagon and hexagon are $\theta_p = 111.38^\circ$ and $\theta_h = 124.31^\circ$ (the latter following from the fact that $\theta_p + 2\theta_h = 360^\circ$), and these differ noticeably from the angles of 108° (for a planar pentagon) and 120° (for a planar hexagon), showing that the differences between spherical and planar geometry are not completely masked in local measurements on a soccer ball.

The truncated icosahedron that underlies a soccer ball also serves as the framework for a molecule of C-60, or "buckyball," which has a carbon atom at each vertex of this polyhedron. It is interesting to contrast buckyball with diamond and graphite, the other two allotropes of carbon. In diamond, each carbon atom occurs at the center of a tetrahedral cage formed by four other carbon atoms, with the angle between neighboring C-C bonds being $\arccos(-1/3) = 109.47^\circ$. In graphite the carbon atoms are arranged in planar hexagonal sheets, with the angle between neighboring C-C bonds being 120°. Buckyball interpolates neatly between these other two allotropes in having bond angles of 108° and 120°.

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Euler's Triangle Inequality via Proofs Without Words

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In November 1983, this MAGAZINE published a special issue [**7**] in tribute to Leonhard Euler (1707–1783) on the occasion of the 200th anniversary of his death. In addition to a number of excellent survey articles, that issue contained a glossary of terms, formulas, equations and theorems that bear Euler's name, the last one of which was the following: