Example. An illustration with three students and two assignments.

$$\left.\begin{array}{l} P_1^1 = \emptyset \\ P_2^1 = \{4\} \\ P_3^1 = \{5,7\} \end{array}\right\} \Rightarrow \begin{array}{l} A_1^1 = \emptyset \\ A_2^1 = \{\emptyset\} \\ A_3^1 = \{\{4\},\emptyset\} \end{array}$$

$$\left.\begin{array}{l} P_1^2 = \emptyset \\ P_2^2 = \emptyset \\ P_3^2 = \{3,9\} \end{array}\right\} \Rightarrow \begin{array}{l} A_1^2 = \{\emptyset\} \\ A_2^2 = \{\emptyset\} \\ A_3^2 = \{\emptyset,\emptyset\} = \{\emptyset\} \end{array}$$

$$\Rightarrow \begin{array}{l} S_1^1 = \left\{A_1^1\right\} = \{\emptyset\} \\ S_2^1 = \left\{A_2^1\right\} = \{\{\emptyset\}\} \\ S_3^1 = \left\{A_3^1\right\} = \{\{\{4\},\emptyset\}\} \\ S_1^2 = \left\{A_1^1, A_1^2\right\} = \{\emptyset, \{\emptyset\}\} \\ S_2^2 = \left\{A_2^1, A_2^2\right\} = \{\{\emptyset\}, \{\emptyset\}\} = \{\{\emptyset\}\} \\ S_3^2 = \left\{A_3^1, A_3^2\right\} = \{\{\{4\},\emptyset\}, \{\emptyset\}\} \end{array}$$

If the professor compiles the set $T^k = \{S_x^k \mid x \text{ is a student in the class}\}$, what can $T^k = \{\{\{\emptyset\}\}\}$ possibly mean? For a class of three or more students, I maintain that each week at most one student failed to have a perfect paper. Can you explain why?

Can you invent a set that might be $\{\{\{\{\emptyset\}\}\}\}$ or $\{\{\{\{\{\emptyset\}\}\}\}\}$? What if the professor has several classes, and the department has several professors?

———o———

Divergence of Series by Rearrangement

Bernard August (mollbern@hotmail.com) and Thomas J. Osler (osler@rowan.edu), Rowan University, Glassboro, NJ 08028

In [1], Michael Ecker proved the divergence of the harmonic series by a novel method, which we will call *divergence by rearrangement.* The main idea was this.

> It is known that a convergent series of positive terms can be rearranged in any way, and the sum remains the same. Suppose we are given a series of positive terms and we *assume it converges*, say to the value S. If upon rearranging the terms we obtain the new sum S' and are able to deduce the contradiction $S' \neq S$, then the series must be divergent.

In this capsule, we use this method to prove the divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n^{r_1}+a_1)^{p_1}(n^{r_2}+a_2)^{p_2}\cdots(n^{r_m}+a_m)^{p_m}}, \tag{1}$$

where all the a_k are nonnegative and all the r_k and p_k are positive with

$$r_1 p_1 + r_2 p_2 + \cdots + r_m p_m \leq 1.$$

Note that special cases of (1) include the harmonic series, the p series $\sum \frac{1}{n^p}$ for $0 < p \leq 1$, and such series as $\sum \frac{1}{\sqrt{n(n+1)}}$ and $\sum \frac{1}{\sqrt[3]{n^2+1}\sqrt[3]{n+2}}$.

To simplify the exposition, we consider the case in which only two factors appear. The reader will have no difficulty seeing how to generalize this proof to include m factors in series (1).

Theorem. Let a and b be nonnegative, and let r, s, p, q be positive with $rp + sq \leq 1$. Then the series

$$\sum_{n=1}^{\infty} \frac{1}{(n^r + a)^p (n^s + b)^q} \tag{2}$$

diverges.

Proof. Assume that the series converges to the value S. Then $S = S_{\text{odd}} + S_{\text{even}}$, where

$$S = \frac{1}{(1^r + a)^p (1^s + b)^q} + \frac{1}{(2^r + a)^p (2^s + b)^q} + \frac{1}{(3^r + a)^p (3^s + b)^q} + \cdots,$$

$$S_{\text{odd}} = \frac{1}{(1^r + a)^p (1^s + b)^q} + \frac{1}{(3^r + a)^p (3^s + b)^q} + \frac{1}{(5^r + a)^p (5^s + b)^q} + \cdots,$$

$$S_{\text{even}} = \frac{1}{(2^r + a)^p (2^s + b)^q} + \frac{1}{(4^r + a)^p (4^s + b)^q} + \frac{1}{(6^r + a)^p (6^s + b)^q} + \cdots.$$

It is clear by comparing corresponding terms that

$$S_{\text{odd}} > S_{\text{even}}. \tag{3}$$

Observe also that

$$S_{\text{even}} = \frac{1}{2^{rp+sq}} \text{ times}$$

$$\left[\frac{1}{\left(1^r + \frac{a}{2^r}\right)^p \left(1^s + \frac{b}{2^s}\right)^q} + \frac{1}{\left(2^r + \frac{a}{2^r}\right)^p \left(2^s + \frac{b}{2^s}\right)^q} + \frac{1}{\left(3^r + \frac{a}{2^r}\right)^p \left(3^s + \frac{b}{2^s}\right)^q} + \cdots\right]$$

Let S^* denote the series above in square brackets. Thus,

$$S_{\text{even}} = \frac{1}{2^{rp+sq}} S^*.$$

Since $rp + sq \leq 1$, we have $\frac{1}{2^{rp+sq}} \geq \frac{1}{2}$. So

$$S_{\text{even}} \geq \frac{1}{2} S^*. \tag{4}$$

Comparing S^* with S term by term, we have $S^* \geq S$. Therefore,

$$S_{\text{even}} \geq \frac{1}{2} S^* \geq \frac{1}{2} S. \tag{5}$$

From (3) and (5), we now have $S = S_{\text{odd}} + S_{\text{even}} > S_{\text{even}} + S_{\text{even}} \geq S$. Thus, we arrive at the contradiction $S > S$ and thereby prove the theorem.

Reference

1. Michael W. Ecker, Divergence of the harmonic series by rearrangement, *The College Mathematics Journal* **28** (1997) 209–210.