Trapezoidal Numbers

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A strikingly simple but little-known result in elementary number theory has recently been discovered as a consequence of the musings of a composer on the Theme of Schoenberg's Variations for Orchestra, Op. 31. He began with the observation that its 12-tone set is partitioned into subsets containing three, four, and five tones. This partitioning into consecutive positive integers determines such thematic details as the number of pitches in a motive, the number of notes in a chord, and the number of measures in a phrase. For Schoenberg's own comments on this piece see his essay [**5**].

From there the composer turned his attention to tone sets containing some number of tones other than twelve and to the feasibility of partitioning each of these into subsets in a similar manner. The question he asked was this: *which numbers can be written as a sum of consecutive positive integers*? The composer (Gamer) then proceeded to check all numbers up to 100 by hand and a remarkable pattern emerged: the only numbers that cannot be written as a consecutive sum are the powers of 2. Incidentally, part of this conclusion appears as problem 78 on page 202 of [**2**].

One approach to answering the question is to realize that we have a natural generalization of **triangular numbers**: the nth triangular number is $T_n = 1 + 2 + \cdots + n$, a consecutive sum that begins with 1. A well-known formula is $T_n = \frac{1}{2}n(n+1)$. A number will be called **trapezoidal** if it is the sum of at least two consecutive positive integers. FIGURE 1 illustrates not only that $12 = 3 + 4 + 5$ is trapezoidal but also that it is the difference of two non-consecutive triangular numbers.

PROPOSITION 1. *All positive integers except the powers of* 2 *are trapezoidal.*

Proof. Let n be a trapezoidal number that is a sum of l consecutive numbers beginning with $k+1$. Then

$$\begin{aligned} n &= (k+1) + (k+2) + \cdots + (k+l) \\ &= T_{k+l} - T_k \\ &= \frac{(k+l)(k+l+1)}{2} - \frac{k(k+1)}{2} \\ &= \frac{l(2k+l+1)}{2}. \end{aligned}$$

Now, one of l or $2k+l+1$ is odd (and the other is even). Therefore, if n is trapezoidal, then it is not a power of 2.

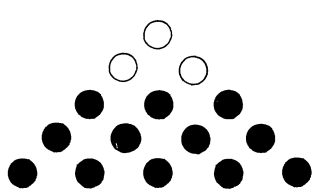

FIGURE 1. $\mathbf{12 = 3 + 4 + 5 = 15 - 3 = T_5 - T_2}$.

Conversely, let n be a positive integer with an odd factor; we show that it is trapezoidal. Since n has an odd factor, so does $2n$, and we can write

$$2n = f_1 f_2$$

where one of f_1 or f_2 is odd and $1 < f_1 < f_2$. To express n as a sum of l consecutive integers beginning with $k+1$, we simply let

$$l = f_1 \quad \text{and} \quad k = \frac{f_2 - f_1 - 1}{2}.$$

Then, $f_2 = 2k + l + 1$ so that

$$\begin{aligned} n &= \frac{f_1 f_2}{2} = \frac{l(2k+l+1)}{2} = T_{k+l} - T_k \\ &= (k+1) + (k+2) + \cdots + (k+l). \end{aligned}$$

Since we get different values of l and k for different choices of f_1 and f_2, we can count the number of ways an integer may be expressed as a consecutive sum by counting its non-trivial odd factors. This result also appears in [**3**] and [**4**].

PROPOSITION 2. *Let $n = 2^r p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ be the prime decomposition of an integer n where the p_i are distinct odd primes. Then the number of ways that n can be written as a sum of at least two consecutive positive integers is*

$$\tau\left(\frac{n}{2^r}\right) - 1 = (r_1+1)(r_2+1)\cdots(r_s+1) - 1$$

where $\tau(m)$ is the number of positive divisors of m.

As an illustration, we consider $n = 225$. Since $225 = 3^2 \cdot 5^2$, there are $(2+1)(2+1) - 1 = 8$ ways to write 225 as a consecutive sum. TABLE 1 shows these sums, using the notation in the proof of Proposition 1.

Odd Factor	$f_1(=l)$	f_2	$k = \frac{1}{2}(f_2 - f_1 - 1)$	Trapezoidal sum
3	3	150	73	74 + 75 + 76
5	5	90	42	43 + 44 + 45 + 46 + 47
3^2	9	50	20	21 + ⋯ + 29
$3 \cdot 5$	15	30	7	8 + ⋯ + 22
5^2	18	25	3	4 + ⋯ + 21
$3^2 \cdot 5$	10	45	17	18 + ⋯ + 27
$3 \cdot 5^2$	6	75	34	35 + ⋯ + 40
$3^2 \cdot 5^2$	2	225	111	112 + 113

TABLE 1

Another approach to the question of which numbers are sums of consecutive positive integers is to focus on the *length* of the sum. For example, for $l = 2$ we get $1 + 2 = 3$, $2 + 3 = 5$, $3 + 4 = 7, \ldots$ and so the odd numbers are consecutive sums. For $l = 3$ we get $1 + 2 + 3 = 6$, $2 + 3 + 4 = 9$, $3 + 4 + 5 = 12, \ldots$ and so multiples of 3 are also such sums. In this way, we can easily list the numbers that are sums of consecutive positive integers:

$$\begin{array}{ll} 3,\ 5,\ 7,\ 9, 11, \ldots & (l=2) \\ 6,\ 9, 12, 15, 18, \ldots & (l=3) \\ 10, 14, 18, 22, 26, \ldots & (l=4) \\ 15, 20, 25, 30, 35, \ldots & (l=5) \\ \vdots & \end{array}$$

Since each row begins with a triangular number, we see that a consecutive sum of length l is a number of the form

$$T_l + kl = \frac{l(l+1)}{2} + kl = \frac{l(2k+l+1)}{2},$$

where $l \geqq 2$ and $k \geqq 0$.

The development above shows that a positive integer n is trapezoidal if and only if $2n$ factors non-trivially as a product of an even and an odd integer. Following this same approach we can answer a more general question: *which numbers are sums of finite arithmetic progressions of positive integers*? If we let d stand for the constant difference in such a progression, then such a sum beginning with b and of length l is given by

$$b + (b+d) + \cdots + (b + (l-1)d) = \frac{l(2b + d(l-1))}{2}.$$

We invite the reader to prove the following result.

PROPOSITION 3. *A positive integer n is the sum of a non-trivial arithmetic progression with constant difference d if and only if there is a factorization $2n = f_1 f_2$ with $f_1 > 1$, $f_2 > d(f_1 - 1)$, and either f_2 is even in the case d is even, or f_1 and f_2 have opposite parity when d is odd. The factor f_1 is the number of terms in the progression.*

We began with a musical observation. We conclude with another, this having to do with the overtone series. The trapezoidal numbers denote those partials within the overtone series that are not octave equivalents of the fundamental, that is, those partials whose frequencies are not multiples by a power of 2 of the frequency of the fundamental. As a corollary to this, and in accordance with the definition of trapezoidal numbers, we know that by adding the frequencies of any two or more consecutive partials of the overtone series we can generate another partial that will not be an octave equivalent of the fundamental. (We are thinking here of an extension of the concept of sine-wave frequency modulation; for an explanation see [**1**].) This principle may prove to be of interest in the physics of sound, particularly with regard to the design of new electronic timbres.

References

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[3] Robert Guy, Sums of consecutive integers, Fib. Quart., 20 (1982) 36–38.

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[5] Arnold Schoenberg, Composition with twelve tones, Style and Idea: Selected Writings of Arnold Schoenberg, Leonard Stein, ed., St. Martins Press, New York, 1975, pp. 214–245.

Variations, Op. 31, Theme