Complex Power Series—a Vector Field Visualization

ALAN D. GLUCHOFF
Villanova University
Villanova, PA 19085

How can we visualize the behavior of complex power series and convey to students a clear understanding of convergence and divergence? In this note we describe a visualization technique based on the use of Pólya vector fields, as introduced by Pólya and Latta [3] and further developed by Braden [1], [2].

In [3], Pólya introduced the notion of a vector field representation of a complex function of a complex variable: If $f(x+iy)=u(x,y)+iv(x,y)$, then the associated Pólya field is $F(x,y)=(u(x,y),-v(x,y))$. A plot of the vector field can then be used to visualize the behavior of f. The complex conjugate is used rather than (u,v) itself because the Cauchy-Riemann equations for analyticity of f translate into zero divergence and zero curl for F; thus a nice interpretation of analyticity in vector field terms is available. Of course, the conjugate field preserves all important features of f.

The vector field representation is ideally suited to depict the process of convergence of a sequence of complex functions. Each function f_n is represented by a vector field F_n on the common domain of definition, and convergence of f_n to a limiting function f can be envisioned as a "stabilizing" of the vector fields F_n to a limiting configuration, that of the vector field for f. Convergence at a point is thus represented by a sequence of arrows based at a common point tending toward a limiting arrow. Divergence appears as either an oscillation or a growth in magnitude without bound of a sequence of common-based arrows.

The production and exploration of such sequences of vector fields is easy in an interactive computer system such as *Mathematica*. We describe a program written in the *Mathematica* language and implemented on the Macintosh II that allows construction of such fields for a complex power series. The program displays the partial sums of a series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in vector field form.

The first input to the program is a domain, which can be a disc or an annulus with center z_0 and any radii. The next input is a "tolerance pair" $[a,b]$; only vectors whose magnitudes are between a and b will be shown. Then the user gives the number of annuli into which the display will be divided, and the closed form expression for a_n.

The program computes and displays the fields in succession; vectors are plotted as arrows with bases on circles determined by the subdivision annuli. The lengths of the vectors are scaled logarithmically to avoid messy overlaps; this does not alter the behavior of the sequence. Arrows whose lengths exceed the upper tolerance level are deleted and replaced by dots. Termination of the program leaves us with a field that may be taken as a representation of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. The animation feature of *Mathematica* allows the fields to be presented in an animated sequence, showing the meaning of convergence and divergence dynamically.

Example. We explore the series $\sum_{n=1}^{\infty} n(z-(1+3i))^n$ with domain $D=\{z:|z-(1+3i)|<2\}$, five subdivision annuli, and tolerance pair $[.01,100]$. FIGURES 1–5 show the fields for the 4th, 6th, 7th, 14th, and 48th partial sums. In the animated sequence, the viewer sees the vectors in each of the outer three rings gyrate, grow in length, then vanish as their vector lengths exceed the prescribed tolerance. (Some of this behavior can be observed in the third ring from the center in FIGURES 1–3; note, for example, the arrow in the upper right whose base is marked by a heavy dot.) The inner two rings continue to gyrate, but eventually stabilize to the configuration shown in FIGURE 5. So we see convergence on a disc centered at $1+3i$ having radius somewhere between 0.8 and 1.2. Further runs with different domains and tolerances can more accurately pin down the disc of convergence and show details on how divergence occurs outside this disc and how stability is achieved inside. Animation then makes this come alive!

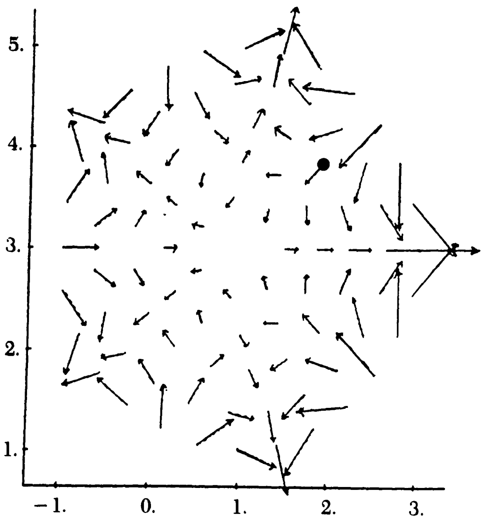

FIGURE 1
4th partial sum of $\sum_{n=1}^{\infty} n(z-(1+3i))^n$ on D.

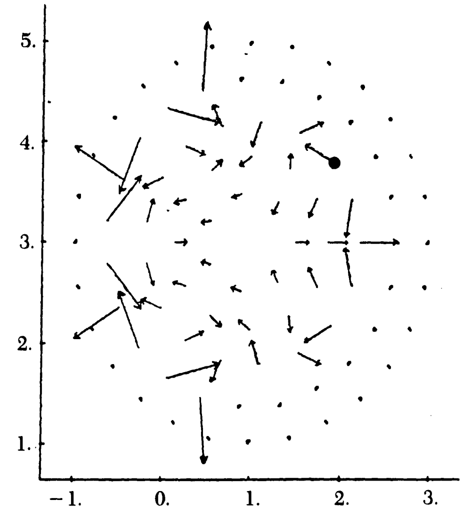

FIGURE 2
6th partial sum.

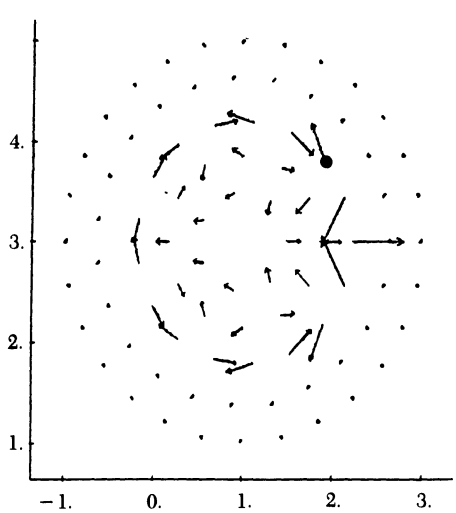

FIGURE 3
7th partial sum.

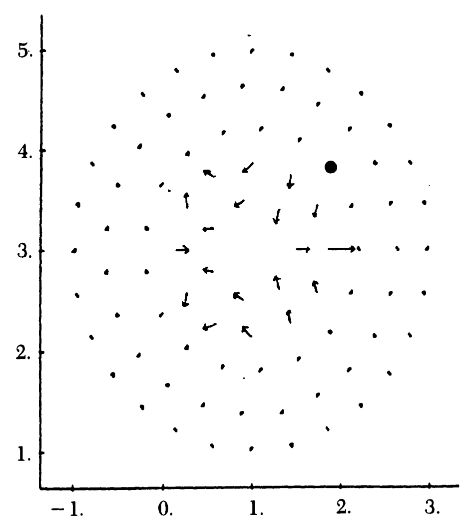

FIGURE 4
14th partial sum.

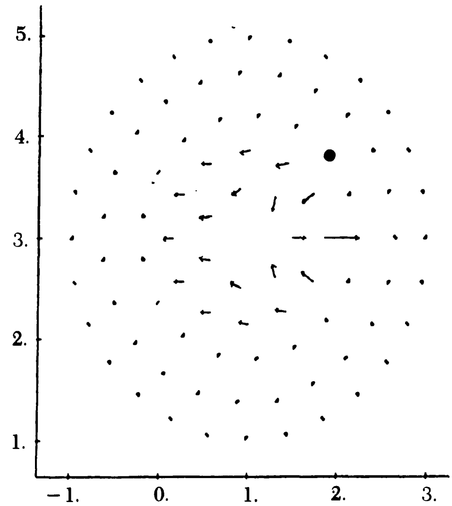

FIGURE 5
48th partial sum.

Even if you are very familiar with the usual treatment of this topic, you may wish to test yourself on the following questions about this example. Answers can be found by looking at the entire run and by making other runs. Try to answer without looking at the figures!

1. On the interior of the disc of convergence, which rings will stabilize first, the inner or outer rings? Why?

2. On any ring on which divergence occurs, will it occur more rapidly (in length) on the right or left portion of the ring? Why?

3. On the rings on which stability occurs, is there more angular gyration on the left or right portion of the ring? Why?

4. If we used a higher upper tolerance, could we see divergence without disappearance of the outer rings, or would they vanish regardless of tolerance used? Why?

5. How would runs for, say, $\sum_{n=1}^{\infty}(z-(1+3i))^n$, compare to our example? Would we see convergence occur more or less rapidly? What about divergence? Why?

A power series is only one of many classes of sequences that can be represented by vector fields; a Laurent series is another obvious candidate. I hope animation complements the standard formal treatment and provides a tangibility that the usual computations, e.g., finding radii of convergence, do not.

I would like to thank Bart Braden for numerous discussions and encouragement in this investigation. I would also like to thank Najib Nadi for technical assistance on the implementation and execution of the program.

REFERENCES

1. Bart Braden, Picturing functions of a complex variable, *College Math. J.* 16 (1985), 63–72.
2. Bart Braden, Pólya's geometric picture of complex contour integrals, this MAGAZINE 60 (1987), 321–327.
3. George Pólya and Gordon Latta, *Complex Variables*, John Wiley & Sons, Inc., New York, 1974.