change doors, and record the frequencies of victories. As you may suspect, the formula $2np(n)^2$, derived from Proposition 1, is not the best way to approximate π; for example, it gives $\pi \approx 3.13$ for $n = 101$. The formula $\frac{1}{1-p(n)}$ for e does much better: if $n = 10$, it gives correctly the first six decimal digits of e.

Finally we notice that, while in Buffon's needle problem the appearance of π is not unexpected (π is the average of the possible inclinations of the needle), a geometrical interpretation of Proposition 1 is not evident.

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A Graph Theoretic Summation of the Cubes of the First *n* Integers

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The complete graph K_{n+1} has $n+1$ vertices and $\binom{n+1}{2}$ edges. Iteratively building the complete graph K_{n+1}, introducing vertices one at a time, and counting new edges incident to each new vertex provides a combinatorial proof that $\sum_{i=1}^{n} i = \binom{n+1}{2}$ [**1**].

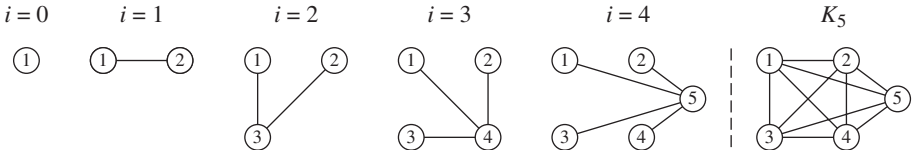

Figure 1 $\sum_{i=1}^{4} i = \binom{4+1}{2}$

Since $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$ it seems natural to look for a combinatorial proof that also uses graphs. The relevant graphs turn out to be *bipartite*, meaning that the vertices are partitioned into two sets and edges occur only between vertices from different parts.

Consider the complete bipartite graph $K_{\binom{n+1}{2},\binom{n+1}{2}}$, which contains $2\binom{n+1}{2}$ vertices and $\binom{n+1}{2}^2$ edges. As before, we will introduce new vertices in n stages and count the new edges as they appear. At stage i, we introduce i new vertices to each side of the graph and count the edges incident to these new vertices. Since $\sum_{i=1}^{n} i = \binom{n+1}{2}$ this process enumerates all the edges in $K_{\binom{n+1}{2},\binom{n+1}{2}}$.

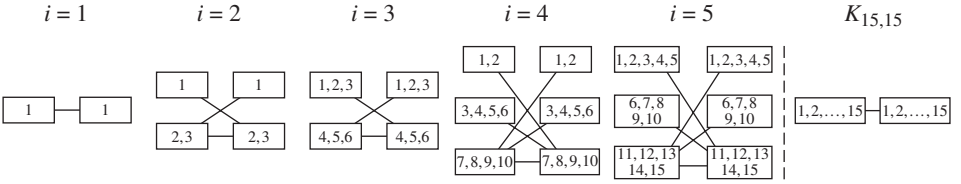

Figure 2 $\sum_{i=1}^{5} i^3 = \binom{5+1}{2}^2$

FIGURE 2 illustrates these stages for $n = 5$. To prevent a deluge of edges in the graph, a complete bipartite graph such as $K_{2,4}$ is represented as $\boxed{1,2}$—$\boxed{1,2,3,4}$. It takes some time to understand the count, so let us walk through the enumeration.

Since the graph is bipartite, new vertices on one side are adjacent only to vertices on the other side. When we consider only the edges among the new vertices, the subgraph $K_{i,i}$ immediately appears, accounting for i^2 edges. It remains to show that these i^2 edges along with the additional edges constructed between new vertices on one side and old vertices on the other side will always total i^3 new edges.

In order to see that we always introduce i^3 new edges at stage i, we partition the new edges into complete bipartite graphs. At stage i, there are already $\binom{i}{2} = \frac{i(i-1)}{2}$ old vertices on each side of the graph. We label the new vertices on each side as $\binom{i}{2}+1, \binom{i}{2}+2, \ldots, \binom{i}{2}+i = \binom{i+1}{2}$. Our method to organize the newly introduced edges into complete bipartite graphs depends upon the parity of i.

When i is odd, the new edges quickly form i disjoint copies of $K_{i,i}$, as follows: For any odd i, we partition the old vertices, which number $\frac{i(i-1)}{2}$, into $\frac{i-1}{2}$ sets of i vertices for each side. Both sets of i new vertices are adjacent to each of the $\frac{i-1}{2}$ sets of i vertices on the other side. This yields $2(\frac{i-1}{2}) = i-1$ additional copies of $K_{i,i}$. Along with the initial copy of $K_{i,i}$ on only the new vertices, we have i copies of $K_{i,i}$ for a total of i^3 new edges.

When i is even, we have to work a bit harder. For even i, we write the number of old vertices as $\frac{i(i-1)}{2} = i(\frac{i}{2}-1)+\frac{i}{2}$. This means we can partition the old vertices on each side into $\frac{i}{2}-1$ sets of i vertices and one set of $\frac{i}{2}$ vertices. This yields $2\left(\frac{i}{2}-1\right)$ copies of $K_{i,i}$ and two copies of $K_{\frac{i}{2},i}$ for $2\left(\frac{i}{2}-1\right)i^2 + 2\frac{i}{2}i = i^3 - i^2$ edges. As before, with the original $K_{i,i}$ between the sets of new vertices, the total once again is i^3 new edges.

Since in either case there are i^3 new edges, this demonstrates that $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$.

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