can be directed to investigate birthday scenarios involving leap years, having birthdays fall within a range of dates, or choosing more than one birthday. Considering this last extension with k people having one birthday and ℓ having a different birthday provides an example with a multinomial distribution.

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A Geometric Look at Sequences that Converge to Euler's Constant

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In this article, we will generate and investigate sequences that converge to Euler's constant, γ. What is novel here is the elementary yet careful attention to the geometric descriptions of the terms of the sequences, allowing us to obtain a convergence rate of order $1/n^2$. Generally, this improved rate can be obtained only with a more cumbersome analytical analysis such as shown in **[2]**, **[3]**, and **[4]**.

Euler's constant is most often defined by comparing the natural logarithm, $\ln(n+1)$, with the nth partial sum

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

of the harmonic series $\sum_{k=1}^{\infty} 1/k$. As shown in Figure 1, h_n can be viewed as the sum of the areas of n rectangles of unit width and heights $1, 1/2, \ldots, 1/n$, and $\ln(n+1)$ is the area under the curve $y = 1/x$ over the interval $[1, n+1]$.

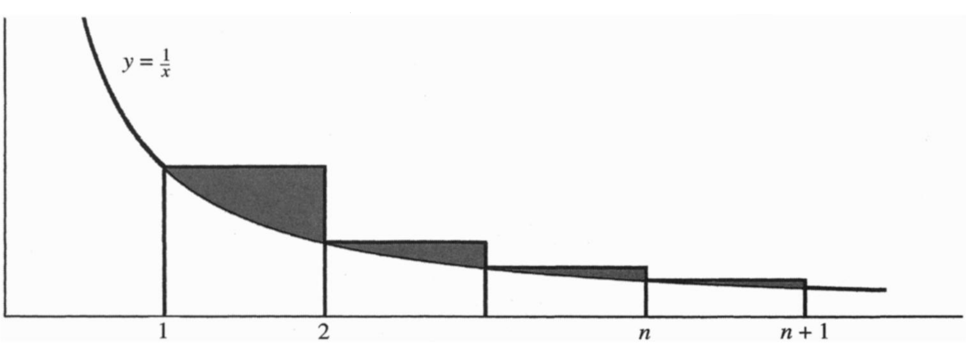

Figure 1.

The difference in these areas,

$$v_n = h_n - \ln(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1),$$

is the sum of the areas of the n curved-bottom regions shown in Figure 1. These regions all have unit width, so they can be translated horizontally to partially fill a unit square, as shown by the darkly shaded regions in Figure 2. If the area of the $1/(n+1)$-by-1 rectangle at the bottom of the unit square is added to v_n, we see that the quantity

$$w_{n+1} = v_n + \frac{1}{n+1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1} - \ln(n+1)$$

is the area of all of the $n+1$ shaded regions indicated in Figure 2.

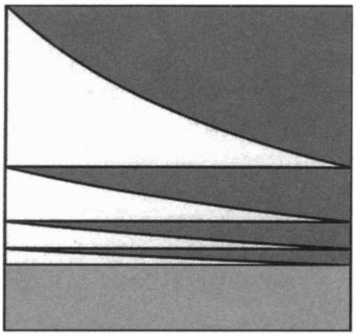

Figure 2.

From this figure, it is geometrically clear that

$$v_{n-1} < v_n < v_n + \frac{1}{n+1} = w_{n+1} < w_n.$$

Thus, the two sequences v_n and w_n converge monotonically in opposite directions to a common limit, γ. That is,

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)\right) \\ &= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right). \qquad (1)\end{aligned}$$

The number $\gamma = 0.57721\ldots$ is called *Euler's constant*, in honor of Leonard Euler (1707–1783), who seems to have been the first to discover it, around 1734. Although γ trails π and e in the famous constants list, this "third constant" has numerous applications in many areas of mathematics, including analysis, probability, special functions, and number theory. An interesting history and selection of examples in which it arises can be found in **[1]**, and Havil's book **[4]** offers a very complete treatment.

Since

$$w_{n+1} - v_n = \frac{1}{n+1},$$

the sequences in (1) converge to γ like $1/n$, a painfully slow rate. To find more rapidly convergent sequences, we modify Figure 1 to include the values of the function $y = 1/x$ at the half-integer values $3/2, 5/2, \ldots, n + 1/2$, as shown in Figure 3.

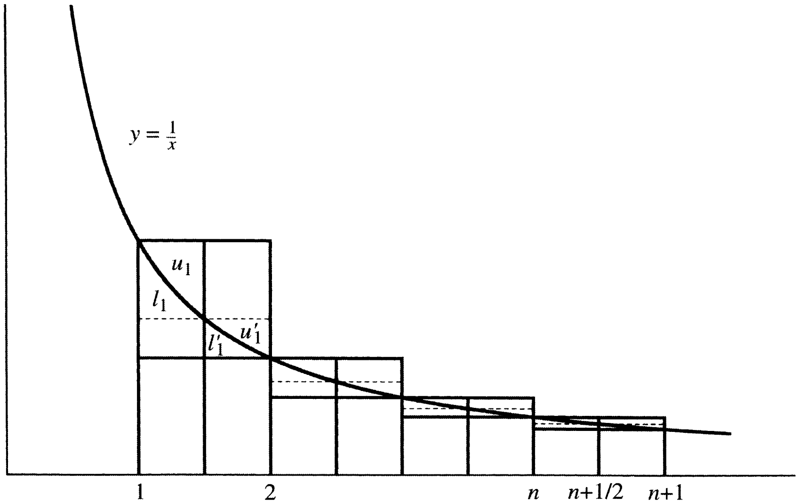

Figure 3.

It is clear geometrically (and can easily be proved analytically) that

$$u_k > l_k > u_k' > l_k' > u_{k+1}. \tag{2}$$

Here u_1 and u_1' are the areas of the two "upper" regions within the first rectangle as labeled in Figure 3, and l_1 and l_1' are the areas of the two corresponding "lower" regions. Likewise, the areas of the upper and lower regions in the kth rectangle are denoted by u_k, u_k', l_k, and l_k'.

We now define the sequences p_n and q_n by

$$p_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1) + \frac{1}{2(n+1)}$$

and

$$q_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right). \tag{3}$$

These sequences represent the respective shaded areas shown in Figure 4.

Since $p_n = q_n - l_n'$, we see immediately that $p_n < q_n$. Similarly, $q_{n+1} = q_n + u_{n+1} - l_n'$, so $q_{n+1} < q_n$, since from (2) we have $l_n' > u_{n+1}$. We can also see that the region of area l_n' is contained in a triangle of width $1/2$ and height

$$\frac{1}{n+1/2} - \frac{1}{n+1} = \frac{1}{(n+1)(2n+1)},$$

so

$$l_n' < \frac{1}{4(n-1)(2n+1)} < \frac{1}{8n^2}.$$

Therefore,

$$q_n - \frac{1}{8n^2} < p_n < q_n.$$

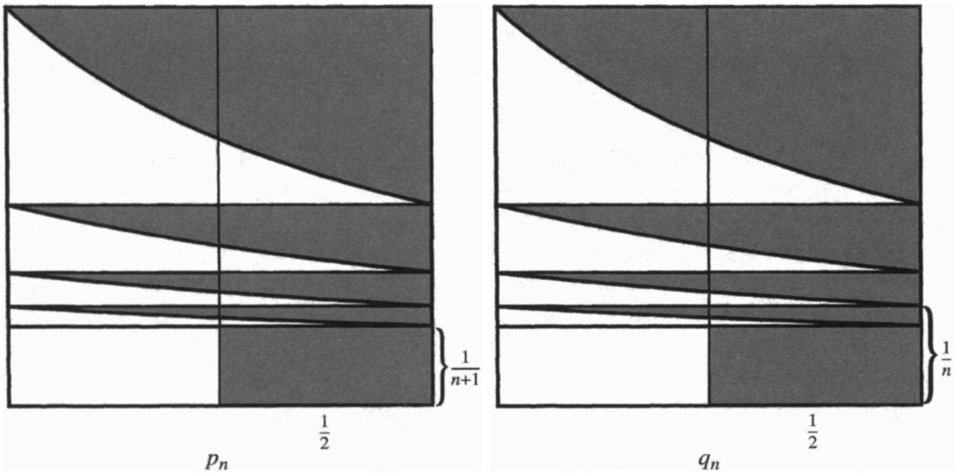

Figure 4.

Altogether, we have shown geometrically that

$$p_n < p_{n+1} < q_{n+1} < q_n < p_n + \frac{1}{8n^2}. \tag{4}$$

That is, the sequences p_n and q_n converge monotonically to γ like $1/n^2$.

The following table of values compares the convergence of the sequences v_n, w_n, p_n, and q_n to Euler's constant $\gamma = 0.57721\ldots$.

n	1	2	3	4	5	6
v_n	0.3069	0.4014	0.447	0.4739	0.4916	0.5041
w_n	1	0.8069	0.7347	0.697	0.6739	0.6582
p_n	0.5569	0.5681	0.572	0.5739	0.5749	0.5755
q_n	0.5945	0.5837	0.5806	0.5793	0.5786	0.5782

Our elementary geometric analysis compares favorably to what can be derived analytically. For example, it can be shown (see **[2]**, **[3]**, and **[4]**) that

$$\frac{1}{24(n+1)^2} < q_n - \gamma < \frac{1}{24n^2},$$

confirming the order $1/n^2$ rate of convergence seen in (4).

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