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Why Are the Gergonne and Soddy Lines Perpendicular? A Synthetic Approach

ZUMING FENG
Phillips Exeter Academy
Exeter, NH 03833
zfeng@exeter.edu

In any scalene triangle the three points of tangency of the incircle together with the three vertices can be used to define three new points which are, remarkably, always collinear. This line is called the Gergonne Line. Moreover cevians through these tangent points are always concurrent at a common point that, together with the incenter, defines a second line, the Soddy Line. Why should these lines be perpendicular? Beauregard and Suryanarayan [**1**] used Eucildean coordinates to establish these results, whereas Oldknow [**2**] used trilinear coordinates. But such a geometric gem deserves a synthetic geometric proof. We shall use the classical theorems of Ceva and Menelaus to define these lines and then establish their perpendicularity by using a certain inversion.

Let ABC be a scalene triangle. Let Ω and I be its incircle and incenter, respectively. Circle Ω touches sides AB, BC, and CA at C_1, A_1, and B_1, respectively. Lines A_1B_1 and AB meet at C_2, and points A_2 and B_2 are defined analogously. As usual, we set $AB = c$, $BC = a$, $CA = b$, and $s = \frac{a+b+c}{2}$. Then it is well known that $AB_1 = AC_1 = s - a$, $BA_1 = BC_1 = s - b$, and $CA_1 = CB_1 = s - c$.

The Gergonne Line. A lovely result for identifying collinear points is Menelaus' theorem:

> Let ABC be a triangle, and let P, Q, R be points on the lines BC, CA, AB, respectively. Then P, Q, R are collinear if and only if
>
> $$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1.$$
>
> (If the lengths are directed, then the product is -1.)

Applying Menelaus' Theorem to line B_1C_1 with triangle ABC (FIGURE 1) yields

$$1 = \frac{AC_1}{C_1B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_1}{B_1A} = \frac{BA_2 \cdot (s-a)(s-c)}{A_2C \cdot (s-b)(s-a)},$$

and so

$$\frac{BA_2}{A_2C} = \frac{s-b}{s-a}.$$

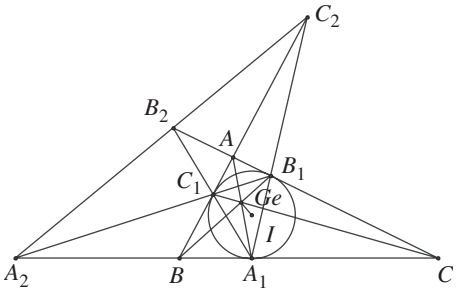

Figure 1

Likewise, we have

$$\frac{CB_2}{B_2A} = \frac{s-c}{s-b} \quad \text{and} \quad \frac{AC_2}{C_2B} = \frac{s-a}{s-c}.$$

Consequently,

$$\frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} \cdot \frac{AC_2}{C_2B} = 1,$$

implying that A_2, B_2, C_2 are collinear, by Menelaus' Theorem. Thus, the line passing through points A_2, B_2, and C_2 is uniquely defined (FIGURE 1). This is the Gergonne Line of triangle ABC.

The Soddy Line. A cevian of a triangle is any segment joining a vertex to a point on the opposite side. We can test cevians for concurrence by using Ceva's Theorem:

> Let AD, BE, CF be three cevians of triangle ABC. Then segments AD, BE, CF are concurrent if and only if
>
> $$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Note that

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = \frac{(s-a)(s-c)(s-b)}{(s-c)(s-b)(s-a)} = 1.$$

By Ceva's Theorem, lines AA_1, BB_1, and CC_1 are concurrent (FIGURE 1). The point of concurrency is the Gergonne point Ge of triangle ABC. Because our triangle is scalene, I does not lie on any of the lines AA_1, BB_1, and CC_1. Hence I and Ge are distinct points. Thus there is a unique line passing through I and Ge (FIGURE 1). This line is the Soddy line of triangle ABC.

The Gergonne line and the Soddy line are perpendicular. We apply a certain inversion to show that $A_2B_2 \perp IGe$. Given a point O in the plane and a real number $r > 0$, the inversion through O with radius r maps every point P (distinct from O) to the point P' on the ray OP such that $OP \cdot OP' = r^2$. We also refer to this map as inversion through γ, the circle with center O and radius r. Key properties of inversion that will be used are (for details, please see [**3**]):

(a) Lines passing through O invert to themselves (though the individual points on the line are not all fixed, FIGURE 2, left).

(b) Lines not passing through O invert to circles through O, and vice versa (FIGURE 2, middle).

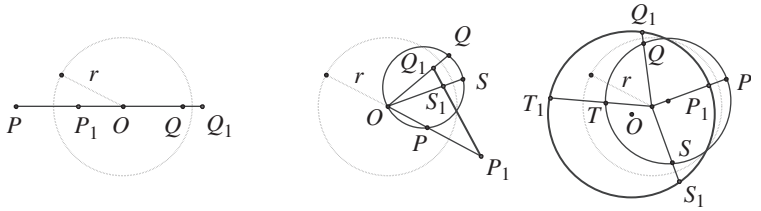

Figure 2

(c) Circles not passing through O invert to circles not passing through O (FIGURE 2, right).

(d) Inversion is a conformal map; that is, inversion preserves the angle between (the tangent lines of) any curves at their intersection points.

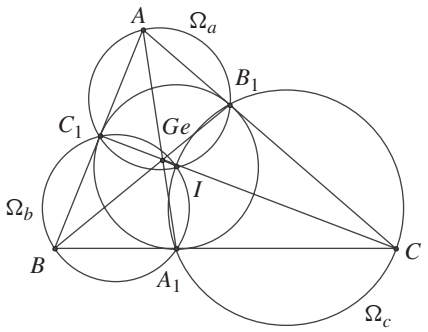

Figure 3

We consider the inversion $\mathbf{I}$ with respect to the incircle Ω (FIGURE 3). Let $\mathbf{I}(P)$ denote the image of element P under the inversion. Then $\mathbf{I}(A_1) = A_1$, $\mathbf{I}(B_1) = B_1$, and $\mathbf{I}(C_1) = C_1$. Because $\angle IB_1A = \angle IC_1A = \frac{\pi}{2}$, points A, B_1, I, C_1 lie on a circle. Let Ω_a denote this circle. We define circles Ω_b and Ω_c (FIGURE 3) analogously. By property (b), $\mathbf{I}(\Omega_a) = B_1C_1$, $\mathbf{I}(\Omega_b) = C_1A_1$, and $\mathbf{I}(\Omega_c) = A_1B_1$.

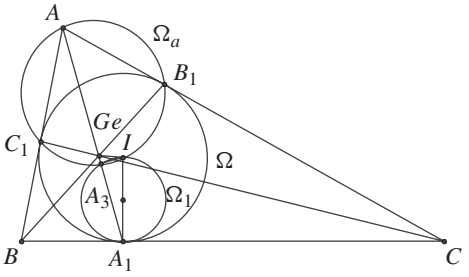

Figure 4

Let Ω_1 be the circle with segment IA_1 as a diameter (FIGURE 4). Then the image of Ω_1 under the inversion is a line, by property (b). Because Ω and Ω_1 are tangent to each other at A_1, by property (d), their images should also be tangent to each other at the image of A_1. It follows that $\mathbf{I}(\Omega_1) = BC$.

Let A_3 be the foot of the perpendicular from I to segment AA_1 (FIGURE 4). Because AI is a diameter of Ω_a and IA_1 is a diameter of Ω_1, A_3 lies on both Ω_a and Ω_1. Thus, $\mathbf{I}(A_3)$ is the intersection of $\mathbf{I}(\Omega_a) = B_1C_1$ and $\mathbf{I}(\Omega_1) = BC$; that is, $\mathbf{I}(A_3) = A_2$ (FIGURE 1).

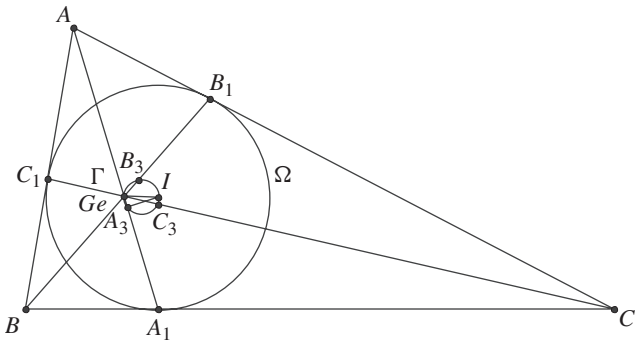

Figure 5

Points B_3 and C_3 are defined analogously and the equations $\mathbf{I}(B_3) = B_2$ and $\mathbf{I}(C_3) = C_2$ follow in similar manner (FIGURE 5). Because $\angle IA_3Ge = \angle IB_3Ge = \angle IC_3Ge = \frac{\pi}{2}$, points A_3, B_3, C_3, I, Ge lie on a circle Γ with IGe as its diameter. By property (b), $\mathbf{I}(\Gamma)$ is a line; that is, points A_2, B_2, C_2 lie on a line. (This is another proof of the existence of the Gergonne line. Furthermore, this shows that $\mathbf{I}(Ge)$ also lies on the Gergonne line.) By property (a), the image of ray IGe is ray IGe. By property (d), to show $A_2B_2 \perp IGe$ (FIGURE 1), it suffices to show that circle Γ and ray IGe are perpendicular at their intersection point Ge . But this is evident, because IGe is a diameter of circle Γ (FIGURE 5).

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Golden Matrix Ring Mod p

KUNG-WEI YANG
Western Michigan University (Retired)
Kalamazoo, MI 49008

Playing with the "golden matrix ring" $\mathbb{Z}[A]$ ($A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$), we had fun proving identities involving Fibonacci numbers ($F_0 = 0$, $F_1 = 1, 1, 2, 3, 5, 8, 13, 21, \ldots$) in [**6**]. Here we return to $\mathbb{Z}[A]$ and show that if we reduce $\mathbb{Z}[A]$ modulo p, then we will get a very neat proof of one of the more remarkable properties of Fibonacci numbers: *every prime p divides some (hence infinitely many) Fibonacci numbers.*