The Classification of Similarities: A New Approach

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Two well-known types of geometric transformations are the isometry and the similarity. A similarity with factor k, $k > 0$, is defined as a bijective transformation f of the plane onto the plane such that for every segment XY the distance between $f(X)$ and $f(Y)$ is k times the distance between X and Y. An isometry is a similarity with $k = 1$. For isometries, we have a well-known classification. Any isometry is one of the following transformations: a translation, a rotation, a reflection or a glide reflection. For similarities this classification is extended as follows: any similarity transformation that is not an isometry is either a dilative rotation or a dilative reflection. The former transformation preserves orientations and is called a direct transformation, whereas the latter changes orientations and is called an opposite transformation. The classification of similarities is not fully proved in most of geometry textbooks. To our knowledge, only [**1**] Chapter 5 and [**2**] give a satisfactory proof of the above classification of similarities. In this paper, an alternative proof is presented based upon Apollonius circles.

For completeness, we recall some definitions. A *central dilatation* or *stretch* with center C and factor k, $k \neq 0$, is a bijective transformation f of the plane onto the plane with C as a fixed point and the property $\overrightarrow{Cf(X)} = k \cdot \overrightarrow{CX}$ for any point X. A *dilative rotation* is the composition of a rotation around a center C through an angle α and a stretch with the same center C and a factor k. A *dilative reflection* is the composition of a reflection in line ℓ and a stretch with center C, where C is on ℓ. The length of a segment XY is denoted by $|XY|$.

The proofs of the classification

By definition dilative reflections and rotations have a fixed point, their center C. Similarities, as defined above, with factor $k \neq 1$ also have a fixed point, but this should be proved. In the 1960s A.L. Steger discovered an elegant and interesting proof of the following theorem: any non-isometric similarity has a fixed point. From this theorem one derives readily using the classification of isometries that any non-isometric similarity is a dilative rotation or a dilative reflection; see [**1**] chapter 5.

Our proof consists of a one-step approach. It is common to state, as in [**1**], that a similarity is determined by its actions on three non-collinear points, but our design starts from a segment with two points. Given two segments A_1B_1 and A_2B_2 of unequal length, a single construction yields two points that are the centers of a dilative reflection and a dilative rotation respectively, both mapping A_1 to A_2 and B_1 to B_2. Consequently, when we take a third point C_1, there are only two possible images for this point.

The proof in the current paper has been found in a classical way, described by Pappus as the method of *analysis*. This means: assume the problem being solved and look for the defining properties of the solution [**4**]. So, given two segments A_1B_1 and A_2B_2 of unequal length and length ratio $1 : k$, we seek all points Z, which are supposed to be the center (fixed point) of a dilatation providing the necessary mapping. Z must have the properties $|ZA_2| = k \cdot |ZA_1|$ and $|ZB_2| = k \cdot |ZB_1|$. Each of these two conditions defines a locus of possible points Z. The loci in question are well-known: they are the so-called Apollonius circles, which we discuss in the next section. The two intersection points of those circles, if they exist, will be taken in consideration as possible centers for the dilatations.

Most of the textbooks lack a satisfactory proof for the fact that any non-isometric opposite similarity is dilative reflection. See among others [**3**, p. 22] and [**5**, p. 43]. As Coxeter stated in [**1**, p. 67]: the direct similarities are treated but opposite similarities seem to have been neglected. Only in [**2**] (after Coxeter made his statement) is a sound proof found for the fact that any opposite similarity is a dilative reflection.

The Apollonius circle

Given two arbitrary points X_1 and X_2 and a constant $k > 0$, $k \neq 1$, the locus of points Z such that $|ZX_2| = k \cdot |ZX_1|$, is a circle Γ, the so-called Apollonius circle for X_1 and X_2 with factor k. See FIGURE 1. The center of Γ lies on the line through X_1 and X_2. Point P is defined as the interior point of X_1X_2 such that $|PX_2| = k \cdot |PX_1|$, whereas P' is defined as the exterior point of X_1X_2 such that $|P'X_2| = k \cdot |P'X_1|$. Hence, both P and P' lie on Γ and PP' is a diameter of Γ.

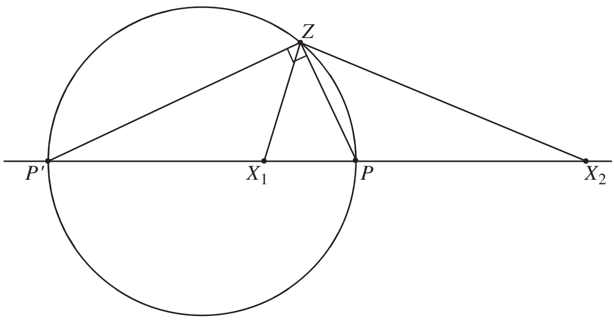

Figure 1 An Apollonius circle

Any point Z on the Apollonius circle Γ satisfies $|ZX_2| = k \cdot |ZX_1|$ and this equality in combination with $|PX_2| = k \cdot |PX_1|$ implies, according to a well-known angle bisector theorem, that ZP is the internal bisector of $\angle X_1ZX_2$. Consequently, for any Z on Γ, there is a dilative reflection with center Z and factor k, which maps X_1 onto X_2. Conversely, if a dilative reflection with center Z and factor k is given that maps X_1 onto X_2, then this center Z lies on Γ. The axis is always ZP.

Likewise, a point Z lies on Γ if and only if it is the center of dilative rotation with factor k transforming X_1 into X_2.

Notice that for any Z on Γ the external bisector of $\angle X_1ZX_2$ is given by ZP', again due to fact that $|P'X_2| = k \cdot |P'X_1|$. The external bisector ZP' and the internal bisector ZP are perpendicular. The aforementioned dilative reflection with center Z, axis ZP, and factor k is identical to the dilative reflection with center Z, axis ZP', and factor $-k$.

Constructing transformations using Apollonius circles

Let two segments A_1B_1 and A_2B_2 be given with $k \neq 1$, where k is defined as $k = |A_2B_2|/|A_1B_1|$. We are looking for a dilative reflection as well as a dilative rotation transforming A_1B_1 into A_2B_2. Let Γ_A and Γ_B denote the Apollonius circles with factor k respectively for the pair A_1, A_2 and the pair B_1, B_2. The possible centers for the dilative reflection and the dilative rotation must lie on the intersection of Γ_A and Γ_B. Unfortunately, we do not know whether these circles intersect. First of all, we shall prove that these circles must intersect.

Let P and P' denote the interior and exterior intersection points of Γ_A with segment A_1A_2. Analogously, the interior and exterior intersection points of Γ_B with segment B_1B_2 are denoted by Q and Q'. The segments PP' and QQ' are diameters of the circles. In FIGURE 2 all these points but not the circles are drawn and two points D and D' are added to produce some similar triangles to be used in the proof. The points D and D' lie on a line parallel to A_1B_1 through B_2 on either side of B_2 at distance $|A_2B_2|$. Since A_1B_1 and A_2B_2 are not parallel, A_2 does not lie on this line. More explicitly we define D and D' by $\overrightarrow{B_2D} = -k \cdot \overrightarrow{B_1A_1}$ and $\overrightarrow{B_2D'} = k \cdot \overrightarrow{B_1A_1}$.

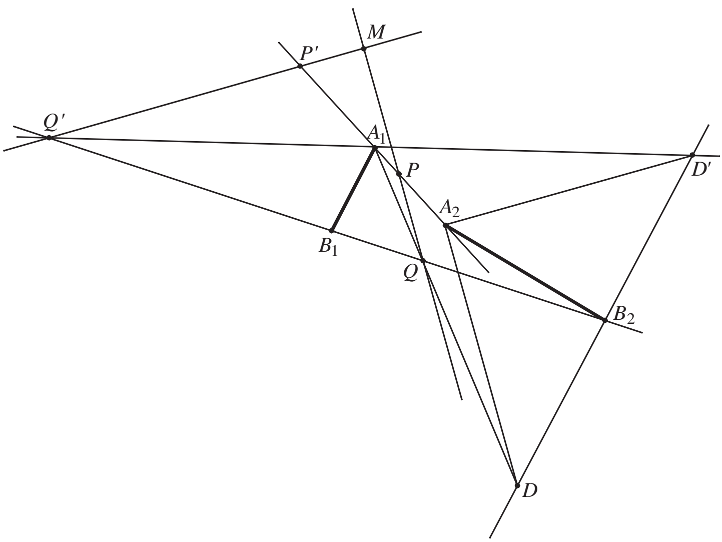

Figure 2 Why are the Apollonius circles intersecting?

Triangle QA_1B_1 is transformed by a stretch with center Q and factor $-k$ into triangle QDB_2 from which we conclude $|QD| = k \cdot |QA_1|$. Now Q is an internal point of segment A_1D, as is P of segment A_1A_2. Remembering $|PA_2| = k \cdot |PA_1|$ we conclude that DA_2 and PQ are parallel. In a similar way using a stretch through Q' with factor k we can prove that $D'A_2$ and $P'Q'$ are parallel.

By the construction of the points D and D' we have the equality $|DB_2| = |A_2B_2| = |D'B_2|$. This implies that $\angle DA_2D' = 90°$. The consequence is that PQ (parallel to DA_2) and $P'Q'$ (parallel to $D'A_2$) are perpendicular, so they surely intersect. The intersection is called M. Since both $\angle PMP'$ and $\angle QMQ'$ are right angles, M lies on Γ_A as well as on Γ_B.

The dilative reflection and the dilative rotation

The dilative reflection defined by factor k, axis MP and center M maps A_1 onto A_2 and the dilative reflection with factor k, axis MQ and center M maps B_1 onto B_2. Since M,

P and Q are collinear, these two transformations are identical. So, the desired dilative reflection has been found. In the next paragraph we focus on the dilative rotation.

In general, Γ_A and Γ_B have besides M another common point, which we call N. A dilative rotation with center N and factor k transforms A_1 onto A_2. Likewise, N is the center of dilative rotation with factor k mapping B_1 onto B_2. These transformations are identical, if $\angle A_1NA_2 = \angle B_1NB_2$. We show that this is indeed the case.

In FIGURE 3, $\angle PMN = \angle PP'N$ and $\angle QMN = \angle QQ'N$, due to the fact that angles inscribed in the same arc of a circle are equal. Since $\angle PMN$ is identical to $\angle QMN$, we conclude $\angle PP'N = \angle QQ'N$. The angles $\angle PNP'$ and $\angle QNQ'$ are right. It follows that the right triangles $PP'N$ and $QQ'N$ are similar. The points A_1 and A_2 divide PP' in the same proportions as B_1 and B_2 divide QQ'. Consequently, figure $PP'NA_1A_2$ is similar to figure $QQ'NB_1B_2$. This implies $\angle A_1NA_2 = \angle B_1NB_2$.

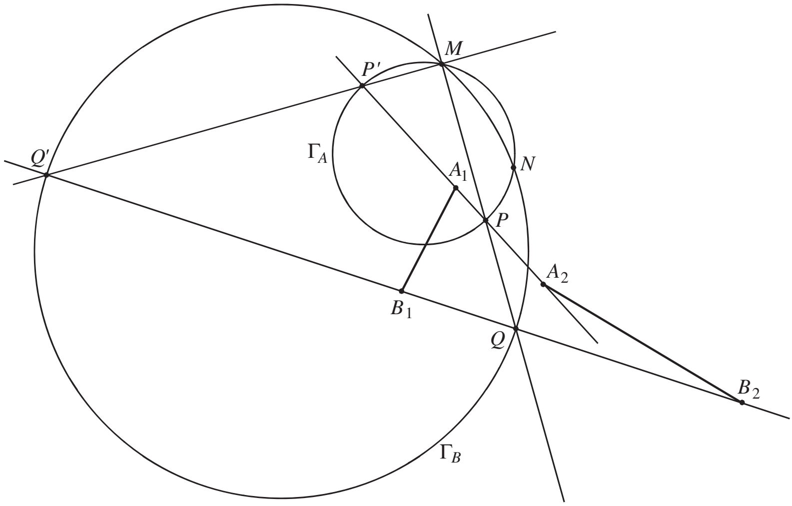

Figure 3 N is the center of the dilative rotation

If Γ_A and Γ_B are tangent in $M = N$, a slightly different derivation applies. We replace in the above derivation $\angle PMN$ and $\angle QMN$ with the angle between PQM and the common tangent line in $M = N$. Similarly to above, we can derive that $\angle PP'N = \angle QQ'N$. As a result of this equality, PP' or A_1A_2 is parallel to QQ' or B_1B_2. The dilative rotation reduces to a central dilatation from $M = N$.

A particular case holds when $P = Q$ or $P' = Q'$. In that case A_1B_1 is parallel to A_2B_2. If $P = Q$, then $P'Q'$, A_1B_1 and A_2B_2 are parallel and we give the line through $P = Q$ perpendicular to $P'Q'$ the role of PQ in the above proof. Then M is again an intersection point of Γ_A and Γ_B. The point given by $P = Q$ plays the role of N. The dilative rotation is a central dilatation.

The case $P' = Q'$ can be handled analogously. Notice that the situation with $P = Q$ and simultaneously $P' = Q'$ cannot happen.

The classification of the similarities

The orientation of a triple (A, B, C) of non-collinear points is either clockwise or counterclockwise according as the traversal A to B to C and back to A is clockwise or not. Let two triangles $A_1B_1C_1$ and $A_2B_2C_2$ be given, such that the lengths of the sides in the latter are k times the lengths in the former, $k \neq 1$. We have shown that there is a unique dilative rotation as well as a unique dilative reflection transforming A_1B_1

into A_2B_2. If the orientations of the triples (A_1, B_1, C_1) and (A_2, B_2, C_2) are the same, the dilative rotation also transforms C_1 into C_2. If the orientations differ, the dilative reflection does so.

As mentioned earlier, any isometry is one of the four transformations: reflection, glide reflection, rotation, or translation. Since a reflection and a rotation are special cases of a dilative reflection and a dilative rotation respectively, we conclude that any similarity is one of the following four transformations: a translation, a dilative rotation, a dilative reflection, or a glide reflection.

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Perfect Matchings, Catalan Numbers, and Pascal's Triangle

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We wish to present a simple combinatorial proof of a determinant formula connecting the Catalan numbers and a matrix derived from Pascal's triangle. We prove the formula by counting perfect matchings in a suitably chosen class of graphs. Although the proof relies on results and techniques from a narrow area, we still believe that it may be interesting also for readers outside this circle, since Catalan numbers are not a very common finding in (or around) the Pascal triangle. We begin with some preliminaries about benzenoid graphs.

A **benzenoid system** is a connected collection of congruent regular hexagons arranged in a plane in such a way that two hexagons are either completely disjoint or have one common edge. To each benzenoid system we can assign a **benzenoid graph**, taking the vertices of hexagons as the vertices of the graph, and the sides of hexagons as the edges of the graph. The resulting graph is simple, planar, 2-connected, bipartite and all its finite faces are hexagons.

A **perfect matching** in a graph G is a collection M of edges of G such that every vertex of G is incident with exactly one edge from M. The number of different perfect matchings in a graph G we denote by $\Phi(G)$.

The motivation for introducing and studying benzenoid graphs came from theoretical chemistry, where they serve as the mathematical model for benzenoid hydrocarbons, a broad and important class of polycyclic carbon and hydrogen compounds in which carbon atoms are arranged in a plane pattern of rings (or cycles) of length six.