On Length and Curvature

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It is visually obvious that for any positive integer n the graph of the function $f(x)=x^{n+1}$ is longer than the graph of $g(x)=x^n$ over the interval $[0,1]$. This observation is posed as a problem in the first edition of Marsden and Weinstein's calculus book ([1], p. 468). A nice proof can be given using the basic differential geometry of plane curves, which many students encounter in calculus. In this note, we discuss this proof as an application of curvature to a problem where it may not be thought to be the natural tool.

A first attempt to prove the required inequality is likely to be by a comparison of the two arc-length integrals. But this approach is not fruitful, since the arc-length integrand $\sqrt{1+(n+1)^2x^2}$ for f does not always exceed the integrand $\sqrt{1+n^2x^{2n-2}}$ for g. Indeed, the graph of f rises more slowly than that of g near 0 but then catches up before reaching 1. A more subtle comparison is necessary, one that relates the two curves along the normal to one of them.

We wish to make precise the following heuristic argument. The functions f and g have the same values at 0 and 1; f always lies below g on $(0,1)$; so the convexity of g should imply that f is longer. But how does one measure convexity in a useful way? It turns out that the concept of curvature is exactly right for this purpose.

Let $X(s)$ denote the arc-length parametrization of the curve $y=x^n$ for $0 \le s \le s_0$ with $X(0)=(0,0)$ and $X(s_0)=(1,1)$. Recall that this represents the curve as a vector-valued function with derivative $X'(s)$ equal to a unit tangent vector $T(s)$ for each s. Let $N(s)$ denote the unit normal vector oriented at a right angle counterclockwise from $T(s)$. Differentiation of the equation $|X'(s)|=1$ shows that $X''(s)$ is orthogonal to $T(s)$; thus it must be a multiple of $N(s)$. The proportionality factor is the signed curvature $k(s)$, defined as $\phi'(s)$, where $\phi(s)$ is the angle that $T(s)$ makes with the horizontal (see FIGURE 1). In symbols $X''(s)=k(s)N(s)$. A related equation is $N'(s)=-k(s)T(s)$, which follows in the same way from differentiating $|N(s)|=1$. The convexity of the curve is then expressed by the inequality $k(s)>0$. For a nice development of the elementary differential geometry of plane curves using these conventions see, for example, [2, p. 531–535].

We now consider a parametrization of $y=x^{n+1}$ derived from $X(s)$ as follows. Let $Y(s)$ denote the point on the graph of f obtained by intersecting this graph with the

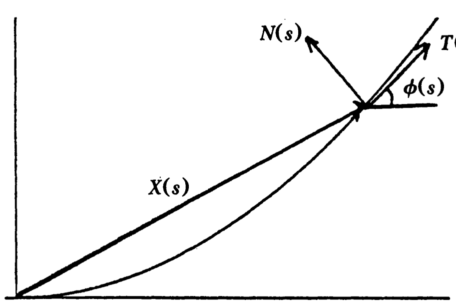

FIGURE 1

normal line to $y=x^n$ at $X(s)$ (see FIGURE 2). Then $Y(s)=X(s)-a(s)N(s)$, where $a(s)$ is a positive function because the graph of f lies below that of g for $0<s<s_0$. Notice that despite the variable name, $Y(s)$ is not an arc-length parametrization of $y=x^{n+1}$, since the length of the tangent vector $Y'(s)$ will generally not be one. Everything else in the equation depends smoothly on s, so the function $a(s)$ must be continuously differentiable. Thus $Y'(s)=X'(s)-a'(s)N(s)-a(s)N'(s)$. But $N'(s) = -k(s)T(s)$ as noted above, so $Y'(s)=(1+a(s)k(s))T(s)-a'(s)N(s)$; whence

$$|Y'(s)| = \sqrt{(1+a(s)k(s))^2 + a'(s)^2}.$$

Thus for all s in $(0, s_0)$ we have that $1 = |X'(s)| < |Y'(s)|$, since both $a(s)$ and $k(s)$ are positive. Recall that the usual arc-length formula holds whether or not the curve is parametrized by arc length. Thus integration gives $\int_0^{s_0}|X'(s)|\,ds < \int_0^{s_0}|Y(s)|\,ds$, which establishes the desired inequality for the lengths of the graphs of f and g.

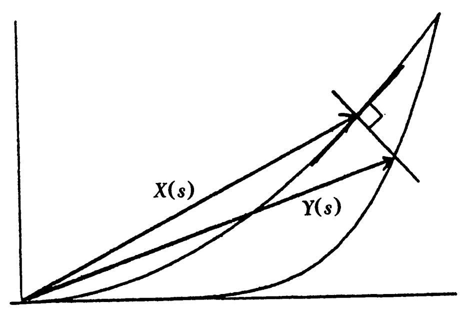

FIGURE 2

This argument generalizes to show that any (rectifiable) curve connecting the endpoints of a smooth convex curve and lying below it will be longer than the original curve. As in the special case above, the convexity implies that $k(s)>0$ and the relative position of the two curves is expressed by the sign of $a(s)$. Of course, the proof works just as well if both $k(s)$ and $a(s)$ are negative, the case of a curve lying above a concave curve.

A key point in this argument that often is not emphasized in a calculus course is that curvature is the intrinsic measure of convexity for a curve, i.e., the measure of how fast the curve is bending away from its tangent line. For the curve $y=g(x)$ the curvature k at x can be computed in terms of derivatives of g by the formula $k(x)=g''(x)/(1+g'(x)^2)^{3/2}$. Notice that it is k that gives real geometric information about the graph of g (though of course the sign of g'' does indicate convexity versus concavity).

We do not see an easy way to relate the lengths of two such curves without using differential geometry. It would be interesting to find a conceptually different approach.

REFERENCES

1. Jerrold Marsden and Alan Weinstein, *Calculus*, Benjamin/Cummings Publishing Co., Menlo Park, CA, 1980.
2. George Simmons, *Calculus with Analytic Geometry*, McGraw Hill Book Co., New York, 1985.