$$D=\begin{vmatrix} * & * & \cdots & * & * & d \\ * & * & \cdots & * & nd & 0 \\ * & * & \cdots & n^2d & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n^{n-1}d & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = (-1)^{\binom{n}{2}} \begin{vmatrix} n^{n-1}d & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & n^2d & 0 & 0 \\ * & * & \cdots & * & nd & 0 \\ * & * & \cdots & * & * & d \end{vmatrix}$$

$$= (-1)^{\binom{n}{2}} n^{1+2+\cdots+n-1}d^n = (-n)^{\binom{n}{2}} d^n.$$

As the signature indicates, it takes $(n-1)+(n-2)+\cdots+2+1=\binom{n}{2}$ switches of rows to bring the determinant to the form in which the upper right triangle consists of zeros. As the students will recall, the value of such a determinant can be calculated by taking the product of the elements in the main diagonal.

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An Efficient Logarithm Algorithm for Calculators

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Dan Kalman and Warren Page [CMJ 16 (January 1985) 57–60] discussed algorithms for raising a number to large or nonintegral powers on calculators equipped with square and square-root keys but having no general power or root key. In this capsule, we will present an algorithm for computing natural logarithms on calculators not equipped with a logarithm key. Initially, we assume that the only keys available are $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, $\boxed{M+}$, $\boxed{M-}$, and $\boxed{MR}$, where the latter three access one memory cell. We will then examine how the algorithm is improved if there is a square-root key. Our intent, as was Kalman and Page's, is to use this as a classroom discussion which combines mathematical applications and algorithmic reasoning.

Integrating the expansion

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1,$$

we obtain

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \qquad |x| < 1. \tag{1}$$

There are two problems with this expansion: it has a small radius of convergence, and it converges very slowly. To obtain a series which converges more rapidly, replace x with $-x$, subtract the two expressions, and rewrite the result as

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots\right] \qquad |x| < 1.$$

Now if we let $N \in Z^+$ and set $x = 1/(2N+1)$, we obtain

$$\ln\left(1+\frac{1}{N}\right) = 2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right].$$

Although this expansion only works for natural numbers N, it can be written recursively as

$$\ln(N+1) = \ln N + \Delta N, \tag{2}$$

where ΔN is the right-hand series in the above equation. This motivates us to

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express any real number x as

$$x = 10^k \times d(1+i), \tag{3}$$

where k is an integer, $d \in \{1,2,\ldots,9\}$, and $|i|$ is a "small" number in the interval $(0,1)$. In general, every number x can be expressed as

$$\begin{aligned} x &= 10^k(\cdot n_1 n_2 \cdots n_k) \\ &= 10^{k-1} n_1 \left(\frac{n_1 \cdot n_2 \cdots n_k}{n_1} \right) \\ &= 10^{k-1} n_1 (1+j) \quad \text{for } j \in (0,1) \end{aligned}$$

as well as

$$\begin{aligned} x &= 10^k(\cdot n_1 n_2 \cdots n_k) \\ &= 10^{k-1}(n_1+1)\left(\frac{n_1 \cdot n_2 \cdots n_k}{n_1+1} \right) \\ &= 10^{k-1}(n_1+1)(1+j') \quad \text{for } j' \in (-1,0). \end{aligned}$$

The best approach is to choose the representation yielding minimum $\{|j|, |j'|\}$. Using the value $i \in \{j, j']$ for which $|i| = \min\{j, |j'|\}$ gives the fastest convergence.

Taking the logarithm of both sides of (3), we have

$$\ln x = k(\ln 10) + \ln d + \ln(1+i). \tag{4}$$

Since $|i|$ is a relatively small number, we need only use a few terms of ln to compute $\ln(1+i)$. We may also use (2) to calculate $\ln 10$ and $\ln d$, by taking $\ln 1 = 0$ as an initial condition. Of course, storing values for $\ln 2$, $\ln 3$, $\ln 5$, $\ln 7$, and $\ln 10 = \ln 2 + \ln 5$ on a card in the calculator case eliminates the need for (2). To seven decimal places, we have:

$$\begin{array}{|c|}\hline \ln 2 = 0.6931472 \\ \ln 3 = 1.0986123 \\ \ln 5 = 1.6094379 \\ \ln 7 = 1.9459101 \\ \ln 10 = 2.3025851 \\ \hline \end{array} \tag{5}$$

For the examples given below, we will be using a calculator with one memory cell. The keys $\boxed{M+}$, $\boxed{M-}$, and $\boxed{MR}$ add to, subtract from, and recall memory, respectively. It will be necessary to raise i to powers to use (1), but we will use the memory to store partial sums, and so it cannot be used in the calculation of powers of i. This will not cause difficulty, however, because we will not need to calculate large powers of i [the series (1) converges rapidly for i sufficiently close to zero]. Hence, to find i^m, we can enter i, press $\boxed{\times}$, and then $\boxed{=}$ $m-1$ times. Our goal in the examples is to find six decimal place accuracy.

Example 1. To compute $\ln 861$, begin with the observation that

$$8.61 = 9(.9566666) = 9(1 - .0433333) \text{ and } 8.61 = 8(1.07625).$$

Then $i = -.0433333$ and $861 = 10^2 \times 9(1 - .0433333)$, and so $\ln 861 = 2\ln 10 + 2\ln 3 + \ln(1 - .0433333)$. To calculate the last term by (1), use the following sequence of operations:

i: enter .0433333, [M−]

$\frac{i^2}{2}$: [×] [=] [÷] [2] [=] gives 0.0009388.

[M−] gives $i - \frac{i^2}{2} = -0.0442721$ in memory.

$\frac{i^3}{3}$: enter .0433333, [×] [=] [=] [÷] [3] [=] gives 0.0000271.

[M+] gives $i - \frac{i^2}{2} + \frac{i^3}{3} = -0.0442992$ in memory.

$\frac{i^4}{4}$: enter .0433333, [×] [=] [=] [=] [÷] [4] [=] gives 0.0000008.

[M−] gives $i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} = -0.0443$ in memory.

$\frac{i^5}{5}$: enter .0433333, [×] [=] [=] [=] [=] [÷] [5] gives 0.

Hence, [MR] shows −.0443, which is ln .9566666 accurate to six decimal places. Therefore,

$$\ln 861 = 2(2.3025851) + 2(1.0986123) - .0443 = 6.7580948,$$

which is accurate to six decimal places.

Example 2. To compute $\ln 507.2143$, write $507.2143 = 10^2 \times 5(1.0144286)$ and $507.2143 = 10^2 \times 6(1 - .1546429)$. Then $i = 0.0144286$, and

$$\frac{i^2}{2} = 0.0001040, \qquad \frac{i^3}{3} = 0.0000010, \qquad \frac{i^4}{4} < 10^{-7},$$

so $i - \frac{i^2}{2} + \frac{i^3}{3} = 0.0143256$ is an approximation of $\ln(1.0144286)$ accurate to seven decimal places. Therefore,

$$\ln 507.2143 = 2\ln 10 + \ln 5 + \ln(1.0144286) = 6.2289337,$$

which is accurate to six decimal places.

Restricting d to being a single digit is hardly necessary, but it minimizes the amount of stored information. One may continue to use only (5) if d is replaced with a quotient p/q of digits—in which case, a smaller value of i may occur in (4).

If the card (5) is to be generated from scratch, it may be obtained more rapidly by using a sequence such as:

$$\begin{aligned}
\ln 2 &= \Delta_1 \\
\ln 8 &= 3\ln 2 \\
\ln 9 &= \ln 8 + \Delta_8 \\
\ln 3 &= \tfrac{1}{2}\ln 9 \\
\ln 10 &= \ln 9 + \Delta_9 \\
\ln 5 &= \ln 10 - \ln 2 \\
\ln 7 &= \left[\ln 10\left(\tfrac{2}{3}\right)(1.05)\right].
\end{aligned}$$

This is because Δ_8 and Δ_9 require only a few terms to calculate.

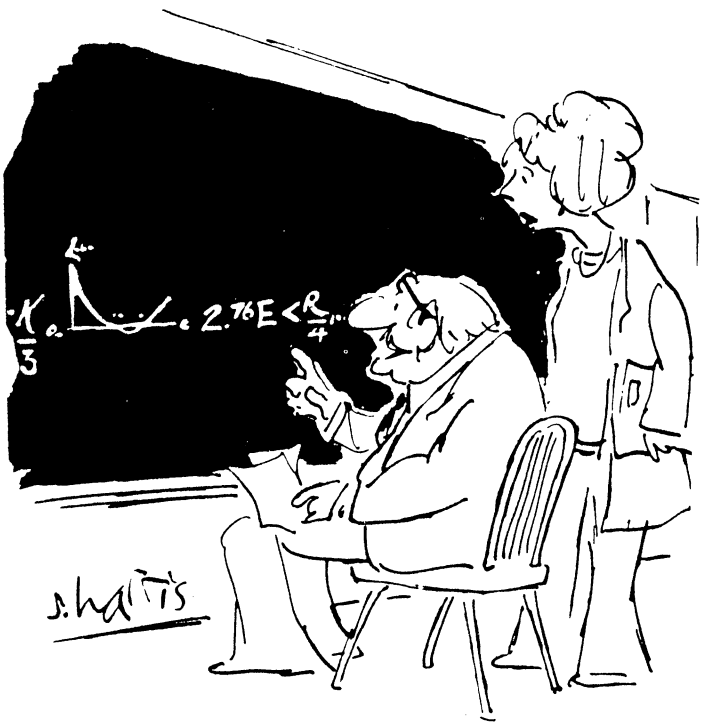
If the calculator has a square root key, one can calculate $\ln(1+i)$ in fewer steps by using

$$\ln(1+i) = 2^k \ln(1+i)^{1/2^k}$$

Example 3. To compute $\ln 16.37$, write $16.37 = 10 \times 2(1 - .1815)$. Then

$$\begin{aligned} \ln(1 - .1815) &= 32 \ln(1 - .1815)^{1/32} \\ &= 32 \ln(1 - .0062394) \\ &= -.2002816 \end{aligned}$$

using just two terms of (1). Note that a large power of 2 cannot be used since there will be excessive roundoff error.



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"The beauty of this is that it is only of theoretical importance, and there is no way it can be of any practical use whatsoever."

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