On the other hand, there are infinitely many *rational* solutions. In fact, if $\beta \in \mathbb{Q}^* = \mathbb{Q} - \{0\}$, then for all $\lambda \in \mathbb{Q}^*$ we have

$$\beta = \left((\lambda^2 + \beta)/2\lambda\right)^2 - \left((\lambda^2 - \beta)/2\lambda\right)^2.$$

For the Goldbach conjecture we need to work in the opposite direction; that is, we consider the values of $x^2 - y^2$ when $x = n$ is fixed and y varies in $\mathbb{Z}$, and we look for appropriate primes p and q.

Whether or not $(*)$ is true, it is intriguing to ask how often $k = 1$ works in $(*)$ and to make the following conjecture:

$(**)$ There are infinitely many integers $n \geq 2$ for which there exist primes p and q such that $n^2 - 1 = pq$.

Statement $(**)$ is clearly equivalent to the twin prime conjecture. Thus, while the Goldbach and twin prime conjectures are not the same, they are evidently facets of the same jewel.

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A Lesser-Known Goldbach Conjecture

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In Example 64 of his recent article [**1**], R. K. Guy asked about the representations of odd positive integers as sums of a prime and twice a square, mentioning that Ruemmler had found the first exception to be 5777 and wondered if it might be the last.

This problem has a long history, beginning with Christian Goldbach, best known for his 1742 conjecture that every even integer can be represented as the sum of two primes. In the last paragraph of a letter to Leonhard Euler dated 18 November 1752, Goldbach expressed his belief that every odd integer could be written in the form $p + 2a^2$, where p is a prime (or 1, then considered a prime) and $a \geqslant 0$ is an integer [**2**, p. 594]. Writing in his usual mixture of German and Latin, Goldbach said:

> Noch habe ein kleines ganz neues theorema beyzufügen, welches so lange vor wahr halte, donec probetur contrarium: Omnis numerus impar est = duplo quadrati + numero primo, sive $2n - 1 = 2aa + p$, ubi a denotet numerum integrum vel 0, p numerum primum, ex. gr. $17 = 2 \cdot 0^2 + 17$, $21 = 2 \cdot 1^2 + 19$, $27 = 2 \cdot 2^2 + 19$, etc.

In his reply dated 16 December 1752, Euler wrote that he had verified this up to 1000 [**2**, p. 596], and in a letter dated 3 April 1753 [**2**, p. 606] he extended the verification up to 2500. Euler also remarked that as the odd number $2n+1$ increased, the number of representations tended to increase.

Over a century later, in 1856, Moritz A. Stern, professor of mathematics at Göttingen, became interested in this problem, perhaps from having read the Goldbach-Euler correspondence [**2**] published by Euler's grandson in 1843. Stern and some of his students checked the odd integers up to 9000 and found two exceptions, 5777 and 5993, which I shall refer to as *Stern numbers* [**3**]. The former of these is the one mentioned in Guy's article [**1**]. Stern also reported that the only primes up to 9000 that could not be expressed in the form $p+2a^2$ in terms of a smaller prime p and an integer $a>0$ were 17, 137, 227, 977, 1187, and 1493, which I shall refer to as *Stern primes*. The tables constructed by Stern and his students were preserved in the library of Adolf Hurwitz, professor of mathematics at Zurich, and made available by Pólya to Hardy and Littlewood [**4**].

I have recently checked these results up to 1,000,000 on a microcomputer. There are no new Stern numbers or Stern primes, only the eight found by Stern. In view of the smallness of the known Stern numbers and Stern primes, it is tempting to replace Goldbach's original conjecture by the new conjectures that there are only a finite number of Stern numbers and Stern primes, and that the list above is complete. The reason for this, of course, is that the average number of ways of expressing an odd integer $2n-1$ as a prime plus twice a square increases without limit as n increases. For example, since the number of primes up to n is on the order of $n/\log n$ while the number of squares up to n is on the order of $n^{1/2}$, the average number of ways of expressing odd numbers up to $3n$ as a prime plus twice a square is at least

$$\frac{n}{\log n}\cdot n^{1/2}\cdot\frac{1}{3n}=\frac{n^{1/2}}{3\log n}$$

("at least" because there are also sums involving primes or squares between n and $3n$). In fact, Hardy and Littlewood [**4**] listed as their Conjecture I: *Every large odd number n is the sum of a prime and the double of a prime. The number $N(n)$ of representations is given asymptotically by*

$$N(n)\sim\frac{\sqrt{2n}}{\log n}\prod_{\varpi=3}^{\infty}\left(1-\frac{1}{\varpi-1}\left(\frac{2n}{\varpi}\right)\right).$$

Here ϖ runs through the odd primes and $\left(\frac{2n}{\varpi}\right)$ is Legendre's symbol. The meaning of the factors in the infinite product can be seen by considering the factor for $\varpi=3$. For $a\equiv 0$, 1, or 2 mod 3 we have $2a^2\equiv 0$, 2, or 2 mod 3. If an odd number $n\equiv 0$ mod 3, then one-third of the time $n-2a^2$ is divisible by 3 and cannot be a prime; in this case $\left(\frac{2n}{3}\right)=0$ and the factor for $\varpi=3$ is 1. But if the odd number $n\equiv 2$ mod 3, then two-thirds of the time $n-2a^2$ is divisible by 3 and cannot be a prime; in this case $\left(\frac{2n}{3}\right)=1$ and the factor for $\varpi=3$ is only $\frac{1}{2}$. Both Stern numbers 5777 and 5993 (and all the Stern primes, as well) are congruent to 2 mod 3, as one would expect. In fact, for both 5777 and 5993 there is a predominance of $+1$ Legendre symbols for small primes: For $\varpi=3,5,7,11,13,\ldots$, $\left(\frac{2n}{\varpi}\right)=1,1,1,1,1,-1,-1,1,-1,\ldots$ for $n=5777$ and $\left(\frac{2n}{\varpi}\right)=1,1,1,-1,0,1,1,1,1,\ldots$ for $n=5993$.

This increase in the average number of representations in the form $p+2a^2$ can be seen in the computer calculations. A consequence of this is that if one lists all the numbers that can be represented in exactly $N\geqslant 0$ ways as a prime and twice a

square, the list appears to be finite for every value of N. Up to 200,000, for example, we find:

2 numbers, 5777 and 5993, that cannot be represented in this way;

28 numbers ranging from 17 to 6,797 that can be represented in exactly one way;

109 numbers ranging from 3 to 59,117 that can be represented in exactly two ways;

225 numbers ranging from 13 to 48,143 that can be represented in exactly three ways;

364 numbers ranging from 19 to 87,677 that can be represented in exactly four ways;

499 numbers ranging from 55 to 148,397 that can be represented in exactly five ways; etc.

As Guy [1] pointed out, it is known that the density of Stern numbers is zero. However, the computer calculations suggest an even stronger conjecture: *For every number $N \geqslant 1$ there are only a finite number of odd integers that cannot be represented as the sum of a prime and twice a square in at least N ways.* This conjecture is also stronger than Conjecture I of Hardy and Littlewood [4].

The computer calculations lead to some new sequences that may be of interest to those who collect quaint and curious sequences. If the conjecture above is correct, we can form a sequence whose nth term is the largest odd number that can be represented as the sum of a prime and twice a square in no more than $n - 1$ ways; this sequence appears to exist and to start:

$$5993, 6797, 59117, 59117, 87677, 87677, \ldots.$$

As expected, these numbers are all congruent to 2 mod 3. Another new sequence that definitely exists is defined to have an nth term that is the smallest odd number that can be represented as the sum of a prime and twice a square in at least n ways:

$$3, 3, 13, 19, 55, 61, 139, 139, 181, 181, 391, 439, 559, 619, 619, 829, 859, 1069, \ldots.$$

As expected, these numbers, after 3 itself, are all congruent to 1 mod 3. From these can be constructed yet more obscure sequences, such as that consisting of those numbers that are a smallest odd number that can be represented as the sum of a prime and twice a square in at least n ways for more than one value of n:

$$3, 139, 181, 619, 2341, 3331, 4189, 4801, 5911, 6319, 8251, 9751, 11311, \ldots.$$

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