CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor.

A Direct Proof that Row Rank Equals Column Rank

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Let A be an $m \times n$ matrix with entries in a field F. The *row rank* of A, denoted $\mathrm{rowrk}(A)$, is the maximum size of a linearly independent subset of the rows of A (viewed as elements of F^n). The *column rank* of A, denoted $\mathrm{colrk}(A)$, is the maximum size of a linearly independent subset of the columns of A (viewed as elements of F^m). A celebrated theorem of linear algebra states that $\mathrm{rowrk}(A) = \mathrm{colrk}(A)$ for all matrices A. This note presents a new elementary derivation of this famed result.

We first recall some of the standard proofs found in the linear algebra literature. Perhaps the most common approach [**9**] involves Gaussian elimination and elementary row and column operations. One shows that the elementary row operations do not alter the row rank or the column rank of A. The same is true for elementary column operations. Using these operations, we can transform A to a matrix with r ones on the main diagonal and zeroes elsewhere. The theorem clearly holds for such a matrix, so it holds for A as well. Another popular computational proof [**2**] sets up systems of linear equations that lead to the inequalities $\mathrm{rowrk}(A) \leq \mathrm{colrk}(A)$ and $\mathrm{colrk}(A) \leq \mathrm{rowrk}(A)$. There are also proofs based on determinants of submatrices [**7**, Th. 2.6.6]. A more abstract method [**4**, §50], [**5**, §3.7, Th. 24] associates a linear transformation T to the matrix A. There is a dual linear transformation T' whose associated matrix is the transpose of A. The theorem then follows from the fact that the range of T has the same dimension as the range of T'. Liebeck [**6**] gave a remarkably short proof for complex matrices. Other proofs, valid for arbitrary fields, have been given by various authors [**1**], [**3**], [**8**], [**10**].

Our new proof requires no computations with determinants, linear systems, or echelon forms, and avoids the abstraction of dual spaces and dual maps. Furthermore, our proof gives more geometric intuition than the others for understanding *why* the result is true. The key idea is to see what happens to the row rank and column rank when we delete a row that can be written as a linear combination of the other rows of the matrix. We call such a row *extraneous*.

Before beginning the proof, we recall two well-known facts.

Fact 1. A linearly independent list of vectors in F^d has size at most d.

Fact 2. Applying an *injective* linear map to each vector in a list of vectors preserves the linear dependence or independence of the list.

Now consider an $m \times n$ matrix A with m rows $v_1, v_2, \ldots, v_m$ in F^n and n columns $w_1, w_2, \ldots, w_n$ in F^m. Suppose first that some row of A, say the kth, is extraneous. Let A' be the matrix obtained by deleting that row. Clearly, the row rank of A' is the same as that of A. But what happens to the column rank? To answer this question, first note that $v_k = \sum_{i \neq k} c_i v_i$ for suitable scalars c_i. This means that each column w_j of A lies in the subspace $W = \{(x_1, \ldots, x_m) \in F^m : x_k = \sum_{i \neq k} c_i x_i\}$. Let $w'_1, w'_2, \ldots, w'_n \in F^{m-1}$ be the columns of A', and let $T : F^m \to F^{m-1}$ be the linear map that erases the kth coordinate. Evidently $w'_j = T(w_j)$ for each j. The restriction $T|_W : W \to F^{m-1}$ is *injective*, since for $x = (x_1, \ldots, x_m) \in W$,

$$T(x) = 0 \Rightarrow (x_i = 0 \quad \forall i \neq k) \Rightarrow \left(x_k = \sum_{i \neq k} c_i x_i = 0 \right) \Rightarrow x = 0.$$

Applying Fact 2, we see that a given subset of the columns of A is linearly independent if and only if the corresponding subset of the columns of A' is linearly independent. It follows that $\operatorname{colrk}(A') = \operatorname{colrk}(A)$.

So far, we have shown that deleting an extraneous row does not affect the row or column rank of A. Interchanging rows and columns in this argument, we see similarly that deleting an extraneous column does not change the row or column rank. By repeatedly deleting extraneous rows and columns, one by one, A is thereby reduced to a $p \times q$ matrix B with no extraneous rows or columns, such that B has the same row rank and column rank as A. The rows of the matrix B constitute a list of p linearly independent elements of F^q, so that $p \leq q$ by Fact 1. Similarly, $q \leq p$. Thus, $\operatorname{rowrk}(A) = \operatorname{rowrk}(B) = p = q = \operatorname{colrk}(B) = \operatorname{colrk}(A)$.

References

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An Elementary Proof of an Oscillation Theorem for Differential Equations

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What can we discover about the zeros of solutions of the differential equation

$$y'' + q(t)y = 0 \tag{1}$$

using graphically appealing arguments based on elementary calculus?