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Math Bite: $\mathbb{Q}$ Is Not Complete

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Define the sequence $\{x_n\}$ in $\mathbb{Q}$ by

$$x_n = \begin{cases} \dfrac{1}{10^n}, & \text{if } n = k^2 \text{ for } k \in \mathbb{N}, \\ -\dfrac{1}{10^n}, & \text{otherwise.} \end{cases}$$

Note that the series $\sum_{n=1}^{\infty} x_n$ converges absolutely:

$$\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{1-\frac{1}{10}} - 1 = \frac{1}{9} \in \mathbb{Q}. \tag{1}$$

Consider the partial sums of $\{x_n\}$, $s_n = \sum_{k=1}^{n} x_k$. For each n, $p \in \mathbb{N}$ we have

$$|s_{n+p} - s_n| \le |x_{n+1}| + \cdots + |x_{n+p}|,$$

from which we can see that $\{s_n\}$ is a Cauchy sequence of rational numbers. Since $\sum_{n=1}^{\infty} x_n$ converges absolutely, its sum does not depend on the way the terms of the series are grouped. Therefore

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \left(\sum_{i=n^2}^{(n+1)^2-1} x_i \right).$$

Some algebraic magic, combining terms in geometric progression, tells us that

$$\begin{aligned}
\sum_{i=n^2}^{(n+1)^2-1} x_i &= \frac{1}{10^{n^2}} - \frac{1}{10^{n^2+1}} - \cdots - \frac{1}{10^{(n+1)^2-1}} \\
&= \frac{10^{(n+1)^2-1-n^2} - (10^{(n+1)^2-1-n^2-1} + \cdots + 10 + 1)}{10^{(n+1)^2-1}} \\
&= \frac{10^{(n+1)^2-1-n^2} - \left(\frac{10^{(n+1)^2-1-n^2}-1}{9}\right)}{10^{(n+1)^2-1}} = \frac{\frac{8\times 10^{(n+1)^2-1-n^2}+1}{9}}{10^{(n+1)^2-1}} \\
&= \frac{\frac{8(10^{(n+1)^2-1-n^2}-1)+9}{9}}{10^{(n+1)^2-1}} = \frac{8\left(\frac{10^{(n+1)^2-1-n^2}-1}{9}\right)+1}{10^{(n+1)^2-1}}
\end{aligned}$$

$$= \frac{8 \times 10^{(n+1)^2-1-n^2-1} + 8 \times 10^{(n+1)^2-1-n^2-2} + \cdots + 8 \times 10 + 9}{10^{(n+1)^2-1}}.$$

Hence

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \left(\frac{0}{10^{n^2}} + \frac{8}{10^{n^2+1}} + \frac{8}{10^{n^2+2}} + \cdots + \frac{8}{10^{(n+1)^2-2}} + \frac{9}{10^{(n+1)^2-1}} \right)$$
$$= 0.089088890888889088888889\ldots. \tag{2}$$

Since a real number is rational if and only if its decimal expansion is periodic [**1**], we see that $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} x_n \notin \mathbb{Q}$. We have a Cauchy sequence in $\mathbb{Q}$ that converges to a number not in $\mathbb{Q}$, and this means that $\mathbb{Q}$ is not complete.

Also it is interesting to note that this series of rational numbers converge absolutely to a rational number (1), but it converges to an irrational number (2).

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A Low-Level Proof of Chebyshev's Pre-Prime Number Theorem

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The Prime Number Theorem (PNT) is a favorite of many mathematicians, for several reasons. At the top of the list is probably the nature of the proof: Heavy Complex Analysis yields Combinatorial Information about an Algebraic Object. The complex analysis is applied to the Riemann zeta function (and its Euler product), and is fairly intricate. Furthermore, there are typically several number theoretic reductions (involving several associated number theoretic functions) along the road.

The impression all this gives is that you can't deduce much from the Euler product for Riemann's zeta function without using complex analysis, and you can't get close to the PNT without using a lot of auxiliary prime-oriented functions. It therefore came as a surprise to me that a "proto" version of the PNT could be derived using little more than sophomore-level real analysis of infinite series and one auxiliary prime function. The purpose of this paper is to present this approach. It is not particularly complicated. (In 1948, Selberg [**10**] gave a calculus-level derivation of the PNT itself. His proof is *not* simple!)

PNT or bust The Prime Number Theorem was first proved by Hadamard and de la Vallée Poussin (independently) in 1896. However, the PNT was conjectured by Legendre long before and, in two papers in the early 1850s, Chebyshev [**2, 3**] essentially showed that the PNT as we know it was the only game in town.