NOTES

Periodic Points of the Open-Tent Function

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Given a function $f : S \to S$, it is of great interest in the field of dynamical systems to figure out which points in the set S are eventually sent back to themselves through repeated applications of f. More precisely, people like to know which points x and which positive integers n have the property that $f^n(x) = x$, where f^n denotes the nth iteration of f. Such a point x is called a *periodic point*, and the smallest such n is called the *prime period* of x. (Note that this does not require that n be a prime number.)

According to a famous theorem by Li and Yorke [**8**], for a continuous function f on a line or a closed interval S, if f has a point of prime period 3, then f has a point of prime period n for every n. This amazing result turns out to be a special case of the even more amazing Sarkovskii theorem [**4**, Ch. 11].

To construct a simple example of a continuous function with a point of prime period 3 on the unit interval, we choose $0 \to 1/2 \to 1 \to 0$ as our 3-cycle and connect the points $(0, 1/2)$, $(1/2, 1)$, and $(1, 0)$ by a piecewise-linear function

$$f(x) = \begin{cases} x + 1/2 & \text{if } 0 \le x < 1/2, \\ 2 - 2x & \text{if } 1/2 \le x \le 1. \end{cases}$$

We call f the *open-tent function*. Its graph is given in FIGURE 1.

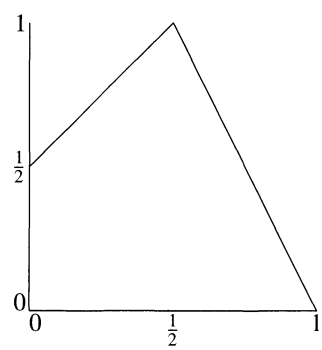

Figure 1 The open-tent function

The open-tent function f is well known and is used as a simple example to illustrate that prime period 3 implies chaos [**1**, p. 248; **3**, p. 135; **6**]. However, knowing the existence of points with various periods and actually finding them are two different matters. In a nice note, David Sprows [**10**] uses binary expansions to construct for the

open-tent function f a point of prime period n for each positive integer n. However, with this method, information about the orbits of f is far from clear. The method we present in this Note gives a far more explicit picture.

The *orbit* of x is the set $\{x, f(x), f^2(x), f^3(x), \ldots\}$. For instance, the orbit of $2/3$ consists of a single point, while the orbit of $1/3$ includes a single additional point, $5/6$. Clearly, the orbit is finite if x is periodic.

Our key for giving a precise description of the orbits is through a method called *encoding* and *decoding*. One might view a binary expansion for any $x \in [0, 1]$ as an identification number, or "ID," for x. Our strategy is to provide each $x \in [0, 1]$ with a different infinite sequence of zeros and ones as its new ID; this is called *encoding*. With this new ID (encoding), we can know the orbital information of the open-tent function f much better. For example, we can tell *how many* points of period n there are and how we can locate *all* of them. We can also locate many other points with interesting orbital features, such as a point that stays obediently in the interval $[1/2, 1]$ for every single iteration of f, escaping exactly once on the one-millionth time.

The open-tent function is an example of a *dynamical system* (S, f), a set S together with a function f from S back to itself. The open-tent example, where the set is $[0, 1]$ would be written $([0, 1], f)$. The key idea of this note is as follows: First, we use a "digital (or symbolic) model" to encode the open-tent function system $([0, 1], f)$, namely, the well-known symbolic dynamical system (G, σ), called the *golden-mean shift*. Then we investigate the encoded orbital information of $([0, 1], f)$ in (G, σ) which is much easier to handle digitally. Finally, we decode the information obtained from (G, σ) back to the system $([0, 1], f)$ in the same way that a CD player decodes its digital codes back into music.

This approach connects many interesting topics in undergraduate mathematics, such as the golden mean, Fibonacci and Lucas numbers, directed graphs, matrices, binary expansions, and coding. Our technique is standard in the field of dynamical systems [**1, 4, 5, 7, 9**], but we provide a rigorous and complete coding algorithm for the open-tent function, including the coding for the numbers of the form $j/2^n$ (the boundary points that arise upon repeatedly bisecting the unit interval), which has previously been unavailable to students.

Unlike the *tent function* (obtained by replacing $x + 1/2$ by $2x$ for $0 \le x < 1/2$ in the definition of f), which is mentioned in almost every dynamical system text and utilizes all 0-1 sequences for its coding, the open-tent function gives us an elementary yet nontrivial example of coding in terms of a proper subset of the set of all 0-1 sequences, as well as a simple yet rich application of symbolic dynamics—a fast-growing branch of modern mathematics [**7, 9**].

The golden-mean shift A *symbolic dynamical system* (X, σ) of the kind considered in this note consists of a set X of infinite sequences of symbols and a shift function σ that knocks off the first term of each sequence. As an example, let $\{0, 1\}$ be the symbol set, and let $X = \{0, 1\}^\infty$ be the set of all infinite 0-1 sequences of the form $x = c_0c_1c_2\cdots$, where $c_i = 0$ or 1. Define the shift function $\sigma : X \to X$ by $\sigma(x) = c_1c_2c_3\cdots$. For the open-tent function, we let G denote the subset of $\{0, 1\}^\infty$ that consists of the sequences in which adjacent zeros are forbidden. The set G together with the shift function σ defined above is called the *golden-mean shift*.

A directed graph H associated with G is shown in FIGURE 2. The vertices of H are the two symbols 0 and 1. The directed edges on H give the rule indicating which symbol can follow another in the sequences of G. Since there is no edge on H from 0 to itself, adjacent zeros are forbidden in the sequences of G. It is easy to see that the elements of G represent all infinite walks on H that start at either of the two vertices

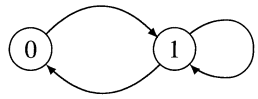

Figure 2 Directed graph H of golden-mean shift

and continue forever. The symbols in the sequence indicate the vertices visited during the walk in the order they are visited. The directed graph H can be recorded by the integer matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

as follows. Let the (i, j) entry, $A(i, j)$, of A be the number of edges from vertex i to vertex j. The matrix A is called the *adjacency matrix* of H. Note that the eigenvalues of A are the golden means

$$\frac{1 \pm \sqrt{5}}{2}.$$

Inductively, we have

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \ldots,$$

$$A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix},$$

where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, are the *Fibonacci numbers*, treated extensively elsewhere in this issue. It is well known that the nth power of an adjacency matrix counts the walks of length n on its graph. In fact, $A^n(i, j)$ counts the walks on H of length n from vertex i to vertex j, and the trace of A^n, $\mathrm{tr}(A^n) = A^n(1, 1) + A^n(2, 2)$, equals the number of closed walks of length n on H. The sequence $\{\mathrm{tr}(A^n)\}_{n=0}^{\infty}$ also satisfies the Fibonacci recurrence relation since $\mathrm{tr}(A^n) = F_{n-1} + F_{n+1}$ with $\mathrm{tr}(A^0) = 2 = L_0$ and $\mathrm{tr}(A^1) = 1 = L_1$. Therefore, $\{\mathrm{tr}(A^n)\}_{n=0}^{\infty}$ is the sequence of famous *Lucas numbers* L_n.

We use the notation $(c_0c_1 \cdots c_{n-1})^\infty$ to indicate the sequence in G or $\{0, 1\}^\infty$ formed by concatenating infinitely many copies of $c_0c_1 \cdots c_{n-1}$. Hence,

$$\sigma((c_0c_1 \cdots c_{n-1})^\infty) = (c_1c_2 \cdots c_{n-1}c_0)^\infty \quad \text{and}$$
$$\sigma^n((c_0c_1 \cdots c_{n-1})^\infty) = (c_0c_1 \cdots c_{n-1})^\infty,$$

so $(c_0c_1 \cdots c_{n-1})^\infty$ has period n under σ. In G, then, we have only one fixed point $1^\infty = 11\cdots$ since $\sigma(1^\infty) = 1^\infty$ and 0^∞ is not in G. We have two points $(01)^\infty$ and $(10)^\infty$ with prime period 2, since $\sigma((01)^\infty) = (10)^\infty$ and $\sigma((10)^\infty) = (01)^\infty$. The element $1^\infty = (11)^\infty$ also has period 2 though its prime period is 1.

The one-to-one correspondence between the elements of G and the infinite walks on H implies

$$\begin{pmatrix} \text{the number} \\ \text{of period-}n \\ \text{points in } G \end{pmatrix} = \begin{pmatrix} \text{the number of} \\ n\text{-step closed} \\ \text{walks in } H \end{pmatrix} = \mathrm{tr}(A^n) = L_n. \tag{1}$$

Encoding and decoding The link between a general dynamical system and its symbolic dynamical system is realized by the encoding and decoding processes. Through them we show that there is a one-to-one correspondence between the period n points of the open-tent function in $[0, 1]$ and those of the golden-mean shift in G. We begin our encoding by making a partition of the unit interval, $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1]$. We then encode every point $x \in [0, 1]$ as an infinite sequence as follows:

$$E(x) = c_0c_1c_2\cdots, \text{ where } c_k = 0 \text{ if } f^k(x) \in I_0 \text{ and } c_k = 1 \text{ if } f^k(x) \in I_1.$$

The sequence $E(x) = c_0c_1c_2\cdots$ is called the *encoding* of x (or the *itinerary* of x). It is the new ID of x. For the open-tent function f, we see that $f(I_0) \subseteq I_1$, so every 0 in the encoding of a number must be followed by a 1, that is, adjacent zeros are forbidden, and $E([0, 1]) \subseteq G$. As an example, since $f^0(0) = 0 \in I_0$, $f(0) = 1/2 \in I_1$, and $f^2(0) = 1 \in I_1$, the first three digits in the encoding of 0 are 011. Since f sends 1 back to 0, the sequence repeats. So, $E(0) = (011)^\infty$. Similarly, $E(1/2) = (110)^\infty$ and $E(1) = (101)^\infty$.

Though a brute force encoding is always possible, decoding is not as straightforward. To make a more general analysis possible, we move back and forth between x and $E(x)$ through the binary expansion of x. Things are complicated a bit by the fact that some rational numbers have two distinct binary expansions. For example, in binary $1/2 = 0.1\bar{0} = 0.0\bar{1}$ just as in decimal $1/2 = 0.5\bar{0} = 0.04\bar{9}$. We must proceed carefully.

Rational numbers of the form $j/2^n$ are called *dyadic numbers*. The dyadic numbers in $(0, 1]$ are exactly the rational numbers in the unit interval that have two distinct binary expansions. If $x \in (0, 1]$ is dyadic, then there exist nonnegative integers j and n such that in lowest terms

$$x = \frac{j}{2^n} = 0.x_1x_2\ldots x_{n-1}1\bar{0} = 0.x_1x_2\ldots x_{n-1}0\bar{1}.$$

Before presenting technical coding formulas, let us see some heuristic descriptions. Because the second piece of f has a slope of -2, an interval in I_1 is stretched by f to double its length, and its orientation is reversed (if $x < y$, then $f(x) > f(y)$), while the first piece of f simply slides an interval in I_0 to the right $1/2$ unit into I_1. The nth iteration of f is a piecewise function that is linear on dyadic intervals of the form $(p/2^n, (p+1)/2^n)$. For the dyadic numbers, we must make the proper choice of binary expansion. A function ψ is defined in (2) to serve this purpose.

Let us consider a generic case where $x \in [0, 1]$ is not dyadic. Suppose $x = 0.x_1x_2x_3\ldots$ is its binary expansion and $E(x) = c_0c_1c_2\cdots$ is its encoding. If $x_1 = 0$, then $x \in I_0$ and $f(x) \in I_1$, so we can determine that c_0c_1 must be 01. Similarly, if $x_1 = 1$, then we can determine that $c_0 = 1$. This is our first step of encoding through the binary expansion. Note that in the first case ($x_1 = 0$), f is applied once to determine the first two symbols in the encoding. In the second case ($x_1 = 1$), f is not applied at all and only the first symbol of the encoding was determined. In both cases, the encoding step ends with a symbol 1. That is, when the orbit enters I_1. To summarize:

$$x_1 = 0 \Rightarrow c_0c_1 = 01,$$
$$x_1 = 1 \Rightarrow c_0 = 1.$$

Having used the first digit of the binary expansion, we ignore it and focus on the second for step 2, because this digit determines the next entry in the encoding, whether it is c_1 or c_2. Suppose $x = 0.*x_2\ldots$. If $x_2 = 0$, then $x \in (0, 1/4)$ or $(1/2, 3/4)$. If the former, then we know that f moved $(0, 1/4)$ onto $(1/2, 3/4)$ without an orientation

reversal in step 1. The next iteration of f sends $(1/2, 3/4)$ to I_1, so the next symbol in the coding is 1 and there is a total of one orientation reversal. Similarly, if $x_2 = 1$, then the next two symbols in the coding are 01 with one orientation reversal, and again, the step ends with the orbit entering I_1. Thus,

$$\begin{aligned} x_2 = 0 &\Rightarrow \text{ the next symbol in the encoding is } 1, \\ x_2 = 1 &\Rightarrow \text{ the next two symbols in the encoding are } 01. \end{aligned}$$

The third step deals with x_3. It produces another orientation reversal, so the orientation is the same as in step 1. Thus,

$$\begin{aligned} x_3 = 0 &\Rightarrow \text{ the next two symbols in the encoding are } 01, \\ x_3 = 1 &\Rightarrow \text{ the next symbol in the encoding is } 1. \end{aligned}$$

The process continues as above through the binary expansion of x. The encoding rules alternate, using x_2 as the template for x_n if n is even, and x_3 if n is odd. The argument, including the subtle handling of the dyadic numbers, is in the proof of Theorem 1. A casual reader could skip the proof.

We now develop technical algorithms for encoding and decoding. The expansions presented are binary. Define $\psi : [0, 1] \to \{0, 1\}^\infty$ by

$$\psi(x) = \begin{cases} x_1x_2\cdots & \text{if } x = 0.x_1x_2\ldots \text{ and is not dyadic, or } 0, \\ x_1x_2\cdots x_{n-1}10^\infty & \text{if } x = 0.x_1x_2\ldots x_{n-1}1\bar{0} \text{ and } n \text{ is odd,} \\ x_1x_2\cdots x_{n-1}01^\infty & \text{if } x = 0.x_1x_2\ldots x_{n-1}0\bar{1} \text{ and } n \text{ is even.} \end{cases} \tag{2}$$

We call $\psi(x)$ the *proper binary expansion* for x. Obviously, ψ is one-to-one and has a left inverse $\phi : \{0, 1\}^\infty \to [0, 1]$ defined by

$$\phi(z_1z_2z_3\cdots) = 0.z_1z_2z_3\ldots = \sum_{k=1}^{\infty} \frac{z_k}{2^k}$$

with $\phi \circ \psi = \mathrm{Id}_{[0,1]}$. It is also clear that ϕ is onto and almost one-to-one—except that it maps two binary expansion sequences to each nonzero dyadic number.

The next function, B, is a bijection between $\{0, 1\}^\infty$ and G. This function and its inverse are at the heart of the encoding and decoding processes, since they provide the correspondence between the proper binary expansion of a point and its encoding. Define $B : \{0, 1\}^\infty \to G$ by $B(z_1z_2z_3\cdots) = y_1y_2y_3\cdots$, where

$$y_n = \begin{cases} 01 & \text{if } n \text{ is odd and } z_n = 0, \\ 1 & \text{if } n \text{ is odd and } z_n = 1, \\ 1 & \text{if } n \text{ is even and } z_n = 0, \\ 01 & \text{if } n \text{ is even and } z_n = 1. \end{cases} \tag{3}$$

For example,

$$\begin{aligned} B(0^\infty) &= B(0000\cdots) = 011011\cdots = (011)^\infty \quad \text{and} \\ B(01^\infty) &= B(0111\cdots) = 0(101)^\infty. \end{aligned}$$

It is easy to see that B is a bijection with the inverse given by $B^{-1}(y_1y_2y_3\cdots) = z_1z_2z_3\cdots$ where $y_1y_2y_3\cdots \in G$, y_n equals 01 or 1, and

$$z_n = \begin{cases} 0 & \text{if } n \text{ is odd and } y_n = 01, \\ 1 & \text{if } n \text{ is odd and } y_n = 1, \\ 1 & \text{if } n \text{ is even and } y_n = 01, \\ 0 & \text{if } n \text{ is even and } y_n = 1. \end{cases} \tag{4}$$

Define the *decoder* $D : G \to [0, 1]$ of E by $D = \phi \circ B^{-1}$. We have the following:

THEOREM 1. *$E = B \circ \psi$, $D \circ E = Id_{[0,1]}$, and $E \circ D|_{E([0,1])} = Id_{E([0,1])}$. In particular, the encoder E is one-to-one and the decoder D is onto.*

Proof. We first show that $E = B \circ \psi$. Suppose $x \in [0, 1]$ is not dyadic and $\psi(x) = x_1x_2x_3\cdots$. In addition, let $E(x) = c_0c_1c_2\cdots = y_1y_2y_3\cdots$, where c_k is 0 or 1 depending on whether $f^k(x)$ is in I_0 or I_1 and y_n is 01 or 1. Suppose further that $B \circ \psi(x) = y_1'y_2'y_3'\cdots$, where y_n' is 01 or 1. We show that $y_n = y_n'$ for all n by induction. If $x_1 = 0$, then $x = f^0(x) \in I_0$ and $f^1(x) \in I_1$, so $c_0 = 0$, $c_1 = 1$, and $y_1 = 01$. On the other hand, by the definition of ψ and (3), $x_1 = 0$ implies $y_1' = 01$. Similarly, if $x_1 = 1$, then $x \in I_1$, so $y_1 = 1$ and $y_1' = 1$. In either case, $y_1 = y_1'$.

Now assume $y_i = y_i'$ for $i = 1, \ldots, n-1$. Since x is not dyadic, there exists an integer j such that $0 \le j \le 2^{n-1} - 1$ and $x \in (j/2^{n-1}, (j+1)/2^{n-1})$, which lies entirely in I_0 or I_1. Each application of f sends such a dyadic interval to another dyadic interval. If the left branch of f is applied, its width remains the same, but if the right branch is applied, the width doubles and the endpoints of the image can both be written with the denominator 2^{n-2}. Such intervals still fall entirely in I_0 or I_1 until they are stretched to a width of 1. Therefore, upon the $(n-1)$st visit of x to I_1, this interval has been stretched to $(1/2, 1)$. If x is in the left half of $(j/2^{n-1}, (j+1)/2^{n-1})$, then $x_n = 0$. If n is even, then upon the $(n-1)$st visit to I_1 the left half of $(j/2^{n-1}, (j+1)/2^{n-1})$ has been stretched to $(1/2, 3/4)$. The next application of f sends the iteration of x already in $(1/2, 3/4)$ to I_1, so $y_n = 1$. But, n even and $x_n = 0$ implies $y_n' = 1$ by (3). Likewise, n odd implies $y_n = 01$ and $y_n' = 01$. In either case $y_n = y_n'$. The parallel argument shows that if x is in the right half of $(j/2^{n-1}, (j+1)/2^{n-1})$, then $x_n = 1$ and $y_n = y_n'$. Therefore, we proved that $y_n = y_n'$. for all n if x is not dyadic.

Suppose $x \in [0, 1]$ is dyadic. It is easy to check that $E(x) = B \circ \psi(x)$ for $x = 0, 1/2$, or 1. If x is dyadic and different from those three, then $x = j/2^n$ in lowest terms with $n \ge 2$, and $x = j/2^n$ is the midpoint of an interval of the form $(p/2^{n-1}, (p+1)/2^{n-1})$ that falls entirely in I_0 or I_1. Let $E(x) = c_0c_1c_2\cdots = y_1y_2y_3\cdots$ as before, and suppose that $B \circ \psi(x) = y_1'y_2'y_3'\cdots$. Since the midpoint of $(p/2^{n-1}, (p+1)/2^{n-1})$ is an element of $(p/2^{n-1}, (p+1)/2^{n-1})$, an argument that parallels the nondyadic case shows that $y_k = y_k'$, but only for $k = 1, 2, \ldots n-1$. Upon the $(n-1)$st visit to I_1, the interval $(p/2^{n-1}, (p+1)/2^{n-1})$ is stretched onto $(1/2, 1)$ and $x = j/2^n$ is mapped to $3/4$. Let q equal the number of applications of f required to produce $n-1$ visits by $(p/2^{n-1}, (p+1)/2^{n-1})$ to I_1, then $c_0c_1c_2\cdots c_q = y_1y_2y_3\cdots y_{n-1}$ with $c_q = 1$. Another application of f sends x to $1/2$, so $c_{q+1} = 1$ and $y_n = 1$. Since $E(1/2) = (110)^\infty$, $E(x) = c_0c_1c_2\cdots c_{q-1}1(110)^\infty = y_1y_2y_3\cdots y_{n-1}(110)^\infty = y_1y_2y_3\cdots y_{n-1}1(101)^\infty$. Thus, if n is odd, $B \circ \psi(x) = B(x_1x_2x_3\cdots x_{n-1}10^\infty) = y_1'y_2'y_3'\cdots y_{n-1}'1(101)^\infty = E(x)$. Similarly, if n is even $B \circ \psi(x) = B(x_1x_2x_3\cdots x_{n-1}01^\infty) = y_1'y_2'y_3'\cdots y_{n-1}'1(101)^\infty = E(x)$.

By construction of the following maps

$$[0,1] \underset{\phi}{\overset{\psi}{\rightleftarrows}} \{0, 1\}^\infty \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G,$$

we get $D \circ E = (\phi \circ B^{-1}) \circ (B \circ \psi) = \phi \circ \psi = \mathrm{Id}_{[0,1]}$. In particular, E is one-to-one and D is onto. Moreover, for any $y \in E([0, 1])$, there is an $x \in [0, 1]$ such that $E(x) = y$, so $E \circ D(y) = E(D(E(x))) = E(x) = y$, thus $E \circ D|_{E([0,1])} = \mathrm{Id}_{E([0,1])}$. This ends the proof. ■

Theorem 1 states that the encoder and the decoder are not inverses of each other, but almost. Hence, we cannot identify the open-tent function with its symbolic model the

golden-mean shift. However, many important dynamical features like periodic points are still in one-to-one correspondence between the two systems.

Using (4) and $D(y) = \sum_{i=1}^{\infty} z_i/2^i$, we can decode any 0-1 sequence in G into a number in [0,1]. The first example is straightforward, but there are a few subtleties of decoding as demonstrated in Examples 2 & 3.

EXAMPLE 1. *To decode the element* $(11011)^\infty$, *use (4) directly to obtain* $D[(11011)^\infty] = 0.\overline{1000} = 2^3/(2^4 - 1) = 8/15$. *So,* $8/15$ *is a point of period* 5.

EXAMPLE 2. *A careless decoding may suggest that* $D[(1101)^\infty] = 0.\overline{100} = 4/7$, *but the odd number of ones in the string* 1101 *tells us that the even-odd parity is switched in the second appearance of* 1101 *in the infinite string* $(1101)^\infty$. *In this case we must list the repeating string twice to get an even number of* 1*s. Thus,* $D[(11011101)^\infty] = 0.\overline{100011} = 5/9$, *and* $5/9$ *has period* 4.

EXAMPLE 3. *What do we do with that final* 0 *when decoding* $(11110)^\infty$? *Simply note that* $(11110)^\infty = 1(11101)^\infty$, *so* $D[(11110)^\infty] = D[1(11101)^\infty] = 0.1\overline{0100} = 19/30$.

EXAMPLE 4. *What is the point* $x \in [0, 1]$ *such that* $f^n(x) \geq 1/2$ *for all* n *except when* n *equals one million? We get the answer by decoding an element of* G *with the right properties:*

$$x = D(1^{1,000,000}01^\infty) = \phi \circ B^{-1}[(11)^{500,000}(01)(11)^\infty] = \phi[(10)^{500,000}0(01)^\infty]$$
$$= \left(\sum_{k=0}^{\infty} \frac{1}{2^{2k+1}}\right) - \frac{1}{2^{10^6+1}} = \frac{2}{3} - \frac{1}{2^{10^6+1}} = \frac{2^{10^6+2} - 1}{3 \cdot 2^{10^6+1}}.$$

The encoder $E : [0, 1] \to G$ is not onto. Let us find exactly which elements of G fall outside $E([0, 1])$. Suppose $w \in G$, but $w \notin E([0, 1])$. Let $x = D(w) = \phi \circ B^{-1}(w)$. Both $B^{-1}(w)$ and $\psi(x)$ are binary expansion sequences of x by the definitions of ϕ and ψ respectively. But $B^{-1}(w) \neq \psi(x)$, for otherwise $w = B(B^{-1}(w)) = B(\psi(x)) = E(x) \in E([0, 1])$. A contradiction. Since ϕ is almost a bijection except for the dual representation of the dyadic numbers,

the elements of G *that fall outside* $E([0, 1])$ *are precisely those elements of* G *that correspond through* B *to the improper binary expansions of the dyadic numbers.*

It is easy to spot these elements. If $y \in G$ begins with a 1, then $\sigma^{-1}(y) = \{0y, 1y\}$, while if y begins with a 0, then $\sigma^{-1}(y) = \{1y\}$. In FIGURE 3 we have an infinite directed graph for $(101)^\infty$ and its preimages. It shows the complete genealogy of the ambiguous sequences in G corresponding to the dyadic numbers. An arrow from w to y indicates $\sigma(w) = y$.

Since the dyadic numbers are the preimages of 1 under f^n for various n and $E(1) = (101)^\infty$, the elements of $E([0, 1])$ that are preimages of $(101)^\infty$ under σ for various n decode to the dyadic numbers. Notice, however, that $\sigma^{-1}[(101)^\infty] = \{(110)^\infty, 0(101)^\infty\}$ even though $f^{-1}(1) = \{1/2\}$. The element $0(101)^\infty \notin E([0, 1])$ and the whole right-hand branch of the directed graph in FIGURE 3 that passes through $0(101)^\infty$ lies outside of $E([0, 1])$.

The necessary and sufficient conditions for $y \in G$ *being in* $E([0, 1])$ *are that either* y *does not have the repeating block* $(101)^\infty$, *or, if it does, then it has a* 1 *just before its repeating block* $(101)^\infty$.

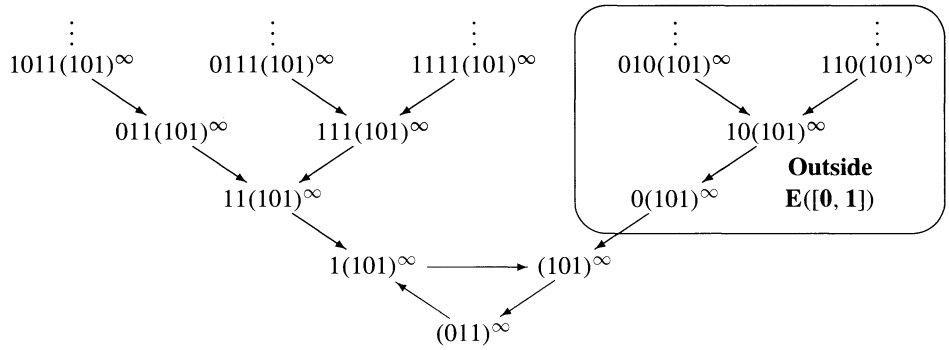

Figure 3 Preimages of $(101)^\infty$ under σ^n

None of the points in FIGURE 3 except the bottom three are periodic though all of their orbits eventually enter a periodic cycle. Such points are called *eventually periodic points*. Now we can describe all the periodic points and eventually periodic points of the open-tent function as our main result.

Periodic points and eventually periodic points

THEOREM 2. *Let f be the open-tent function on $[0, 1]$, and let (G, σ) be the golden-mean shift.*

1. *There is a period-preserving bijection between the set of all periodic points in $([0, 1], f)$ and those in (G, σ). In particular, we can locate all periodic points of f precisely.*
2. *The number of period-n points of f equals the Lucas number L_n.*
3. *The only 3-cycle of f consists of the three dyadic numbers $0 \xrightarrow{f} 1/2 \xrightarrow{f} 1 \xrightarrow{f} 0$. All other dyadic numbers in $[0, 1]$ are eventually period-three points with their orbits eventually ending with the 3-cycle above. The converse is also true.*
4. *A number $x \in [0, 1]$ is a periodic point of f if and only if either*
   - *(a) x is a rational number that can be written as a fraction with an odd denominator (this includes 0 and 1), or*
   - *(b) x is a rational number that can be written in the form $1/2 + j/(2k)$ for some nonnegative integer j and odd positive integer k (this includes $1/2$ and 1).*
5. *A number $x \in [0, 1]$ is a periodic or eventually periodic point of f if and only if x is rational.*

Proof. (1,2). By design, if $E(x) = c_0c_1c_2\cdots$, then $E(f(x)) = c_1c_2c_3\cdots$. Hence, $E \circ f = \sigma \circ E$. This along with the fact that E is one-to-one (Theorem 1) implies that $f^n(x) = x$ if and only if $\sigma^n(E(x)) = E(x)$. Thus, x has period n under f if and only if $E(x)$ has period n under σ. Since none of the elements of G that fall outside $E([0, 1])$ are periodic (FIGURE 3), E serves as a bijection between the period-n points of $([0, 1], f)$ and those of (G, σ). So, the number of period-n points of $[0, 1]$ under f equals the Lucas number L_n by (1). To prove *(3)*, observe that the only prime-period-three elements in G are $(011)^\infty$, $(110)^\infty$, and $(101)^\infty$. They decode to the only prime-period-three elements 0, $1/2$, and 1 respectively in $[0, 1]$. The proof of Theorem 1 implies the rest. We leave the proofs of *(4)* and *(5)* as exercises in the application of decoding. ■

TABLE 1 lists the period-n points of (G, σ) and $([0, 1], f)$ for $n = 1, 2, 3, 4$.

TABLE 1: Periodic points of the open-tent function

Prime Period	$E(x)$	x
1	1^∞	2/3
2	$(01)^\infty$, $(10)^\infty$	1/3, 5/6
3	$(011)^\infty$, $(110)^\infty$, $(101)^\infty$	0, 1/2, 1
4	$(0111)^\infty$, $(1110)^\infty$, $(1101)^\infty$, $(1011)^\infty$,	2/9, 13/18, 5/9, 8/9

Let q_n denote the number of points in $[0, 1]$ having prime period n under f. If $k < n$ and k divides n, then the q_k elements of $[0, 1]$ with prime period k are counted in L_n along with the q_n elements with prime period n. Thus,

$$q_n = L_n - \sum_{k|n, k<n} q_k.$$

With the help of a computer, we calculate some values of q_n in TABLE 2.

TABLE 2: Number of period-n points

n	No. of Period n Pts. L_n	No. of Prime Period n Pts. q_n
1	1	1
2	3	2
4	7	4
5	11	10
10	123	110
20	15,127	15,000
25	167,761	167,750
50	28,143,753,123	28,143,585,250
100	792,070,839,848,373,253,127	792,070,839,820,228,485,000

We should be aware of the limitations of a computer for such a seemingly simple process as calculating the iterations of f at some point x. Try using a spreadsheet or mathematical software that uses floating-point arithmetic to investigate this; you will find that *all orbits* of f end with the 3-cycle $0 \to 1/2 \to 1 \to 0$. Why? The computer uses a finite binary expansion to represent the seed number. In doing so, it has rounded the seed to a dyadic number. By Theorem 2, the orbits of all dyadic points end in that 3-cycle. This phenomenon is quite unique to the open-tent function. It is no longer true if we just move the top of the tent a bit higher or lower! Interested readers may study the orbit diagram (by *Maple*) in FIGURE 4 of the following family of functions with the parameter c:

$$f_c(x) = \begin{cases} cx + 1/2 & x < 1/2 \\ (1+c)(1-x) & x \geq 1/2 \end{cases}, \quad \text{for } -1 \leq c \leq \frac{1+\sqrt{5}}{2}.$$

The orbit diagram plots the parameter c with a gap of 0.02 on the horizontal axis versus the eventual orbit of the critical point $1/2$ under f_c on the vertical axis. A different family that contains the open-tent function is discussed by Bassein **[2]**.

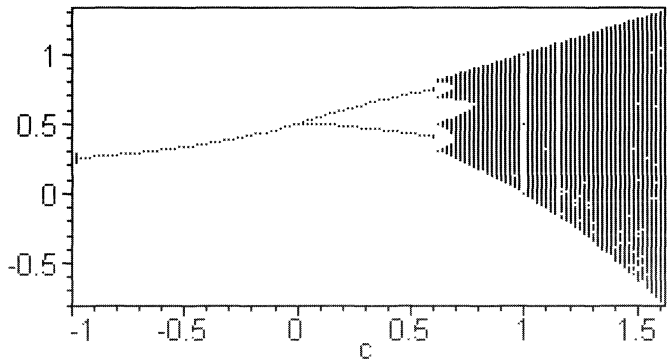

Figure 4 Orbit diagram of $x = 1/2$ under a modified tent function f_c

FIGURE 4 presents a familiar picture of transition to chaos through period-doubling bifurcations [**4**, Ch. 8]. For $-1 < c < 0$, the orbit of $x = 1/2$ tends to an attracting fixed point. At $c = 0$, the family has a period-doubling bifurcation where the attracting fixed point turns into a repelling fixed point and gives birth to an attracting 2-cycle. For $c > 0$, the orbit of $1/2$ tends to an attracting 2-cycle until $c \approx 0.617$ when another period-doubling bifurcation happens that gives birth to an attracting 4-cycle. The dark region of the diagram shows that the orbits of $x = 1/2$ under corresponding f_c are trapped in one or more vertical intervals, jumping back and forth chaotically. When $c = 1$, f_c is the open-tent function, and the orbit of $1/2$ is represented by the three dots that appear on the vertical line $c = 1$. The reason we can see these three dots is not because it is an attracting 3-cycle, but because all numbers are rounded by computer to dyadic numbers that eventually enter the 3-cycle $0 \to 1/2 \to 1 \to 0$ (see FIGURE 3). The open-tent function is unique in this family $\{f_c\}$. For c just off from 1, the orbit of $1/2$ under f_c is chaotic. When $c > (1 + \sqrt{5})/2$, the orbit of $1/2$ escapes, so we see the golden mean one more time to end the orbit diagram! We still do not know how to locate all the periodic points of f_c for all $c \neq 1$ as we do for the open-tent function.

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