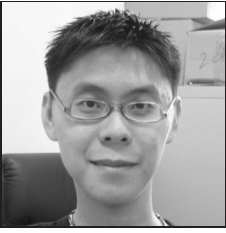# Sums of Consecutive Integers

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Wai Yan Pong (wpong@csudh.edu) received his B.Sc. from the Chinese University of Hong Kong and his M.Sc. and Ph.D. from the University of Illinois at Chicago. He was a Doob Research Assistant Professor at the University of Illinois at Urbana-Champaign for three years. He then moved to California and is now teaching at California State University, Dominguez Hills. His research interests are in model theory and number theory.

While looking for exercises for a number theory class, I recently came across the following question in a book by André Weil [**4**, Question III.4]:

Which natural numbers can be written as the sum of two or more consecutive integers?

The origin of this question is unknown to me, but one can easily believe that it is part of the mathematical folklore. Solutions to this problem and some generalizations may be found in [**1**] and [**2**], for example. We will give a somewhat different proof here, one that we hope readers will find intuitively appealing.

To get a feeling for the problem, let us consider some small values. The list below gives one attempt at expressing the numbers up to 16 as sums of two or more consecutive integers. We leave the right-hand side of the equation blank if there is no such expression for the number on the left-hand side.

$$1 = , \qquad 2 = , \qquad 3 = 1 + 2, \qquad 4 = ,$$
$$5 = 2 + 3, \qquad 6 = 1 + 2 + 3, \qquad 7 = 3 + 4, \qquad 8 = ,$$
$$9 = 4 + 5, \qquad 10 = 1 + 2 + 3 + 4, \qquad 11 = 5 + 6, \qquad 12 = 3 + 4 + 5,$$
$$13 = 6 + 7, \qquad 14 = 2 + 3 + 4 + 5, \qquad 15 = 7 + 8, \qquad 16 = ,$$

A conjecture naturally arises from this list:

Conjecture. *A natural number is a sum of consecutive integers if and only if it is not a power of* 2.

So, is it true? If so, what makes the powers of 2 special?

The Proof

We start by defining a *decomposition* of a natural number n to be a sequence of consecutive natural numbers whose sum is n. The number of terms is called the *length* of the decomposition, and a decomposition of length 1 is called *trivial*. Further, a decomposition is called *odd* (*even*) if its length is odd (even). For example, consider the number 15. It has four decompositions, (15), (7, 8), (4, 5, 6), and (1, 2, 3, 4, 5), one even and three odd.

We will construct a one-to-one correspondence between the odd factors of a number and its decompositions. This proves the conjecture since the powers of 2 are precisely those numbers with only one odd factor (namely 1) and thus they have only the trivial decomposition.

Here is our construction: Let k be an odd factor of n. Then since the sum of the k integers from $-(k-1)/2$ to $(k-1)/2$ is 0, adding n/k to each of them gives the sequence

$$\frac{n}{k} - \frac{k-1}{2}, \quad \frac{n}{k} - \frac{k-1}{2} + 1, \quad \ldots, \quad \frac{n}{k} + \frac{k-1}{2}, \tag{1}$$

whose sum is

$$\sum_{j=-(k-1)/2}^{(k-1)/2} \left(\frac{n}{k} + j\right) = \frac{n}{k} \cdot k \quad + \sum_{j=-(k-1)/2}^{(k-1)/2} j = n + 0 = n.$$

There are now two cases to consider:

(i) $(k-1)/2 < n/k$. Then (1) is already an odd decomposition of n. The length of this decomposition is k.

(ii) $(k-1)/2 \geq n/k$. Then (1) begins with 0 or a negative number. After dropping the 0 and canceling the negative terms with the corresponding positive ones, we are left with the sequence

$$\frac{k-1}{2} - \frac{n}{k} + 1, \quad \frac{k-1}{2} - \frac{n}{k} + 2, \quad \ldots, \quad \frac{n}{k} + \frac{k-1}{2}, \tag{2}$$

which is an even decomposition of n of length $2n/k$.

We call the decomposition in either (1) or (2) the *decomposition of n associated with k*.

To show that every decomposition of n arises this way, suppose $(a+i)_{i=1}^{m}$ is a decomposition of n. Since its sum $m(2a+m+1)/2$ is n, either m or $2a+m+1$ (but not both), depending on the parity of m, is an odd factor of n. It is then straightforward to verify that the given decomposition is the one associated with that odd factor.

We have in fact delivered more than we promised. Note that the condition $(k-1)/2 < n/k$ is equivalent to $k-1 < 2n/k$, but since k is odd the condition is therefore the same as $k < 2n/k$, or equivalently $k < \sqrt{2n}$. Likewise, $(k-1)/2 \geq n/k$ is equivalent to $k > \sqrt{2n}$. Therefore, we have proved the following:

Theorem. *There is a one-to-one correspondence between the odd factors of a natural number n and its decompositions. More precisely, for each odd factor k of n, if $k < \sqrt{2n}$, then (1) is an odd decomposition of n of length k. If $k > \sqrt{2n}$, then (2) is an even decomposition of n of length $2n/k$. Moreover, these are all the decompositions of n.*

An immediate consequence of the theorem is that *the number of decompositions of n is the number of odd factors of n*, which is $\prod_p (e_p + 1)$, where p runs through the odd primes and e_p is the power of p appearing in the prime factorization of n. Let us illustrate both this formula and the theorem with the example $n = 45$. Since $45 = 3^2 \cdot 5$, there should be six decompositions, four odd and two even, and indeed here they are:

Table 1. The decompositions of 45

k	$(k-1)/2$	n/k	decomposition	length	parity
1	0	45	(45)	1	odd
3	1	15	(14, 15, 16)	3	odd
5	2	9	(7, 8, 9, 10, 11)	5	odd
9	4	5	(1, 2, 3, 4, 5, 6, 7, 8, 9)	9	odd
15	7	3	(5, 6, 7, 8, 9, 10)	6	even
45	22	1	(22, 23)	2	even

The Length Spectra

We define the *length spectrum of n*, denoted by $\mathrm{lspec}(n)$, to be the set of lengths of the decompositions of n. According to the theorem, $\mathrm{lspec}(n)$ is the set

$$\left\{k : k \text{ odd}, \ k \mid n, \ k < \sqrt{2n}\right\} \cup \left\{2n/k : k \text{ odd}, \ k \mid n, \ k > \sqrt{2n}\right\}.$$

For example, we have seen that $\mathrm{lspec}(45) = \{1, 3, 5, 9\} \cup \{2, 6\}$ (Table 1). In the following, we record some simple facts about length spectra. Most of them are direct consequences of the theorem. An in-depth treatment of this notion is given in [**3**].

Let $(k_i)_{i=1}^s$ be the list of odd factors of n in ascending order. Thus, $k_1 = 1$, k_2 (if it exists) is the smallest odd prime factor of n, and $n = 2^d k_s$ for some $d \geq 0$. Let r be the largest index $(1 \leq r \leq s)$ such that $k_r < \sqrt{2n}$.

1. The length of an even decomposition of n is of the form $2n/k_j$, and hence the highest power of 2 that divides any even element of $\mathrm{lspec}(n)$ is 2^{d+1}. This observation rules out, for example, the possibility of the set $\{1, 2, 3, 4\}$ being a length spectrum.

2. The smallest number with a length spectrum of size s is 3^{s-1}. The smallest number with m in its length spectrum is clearly

$$1 + 2 + \cdots + m = m(m+1)/2.$$

   In other words, $m \in \mathrm{lspec}(n)$ implies $m(m+1)/2 \leq n$. Hence $m \leq (-1 + \sqrt{1+8n})/2$. This gives an upper bound on the elements of $\mathrm{lspec}(n)$ in terms of n.

3. The longest decomposition of n has length $\max\{k_r, 2n/k_{r+1}\}$ (the maximum is k_r if n has no even decompositions). The shortest non-trivial decomposition of n (if any) has length $\min\{k_2, 2n/k_s\} = \min\{k_2, 2^{d+1}\}$.

4. The condition $k_s < \sqrt{2n}$, or equivalently $k_s < 2^{d+1}$, is clearly both necessary and sufficient for n to have only odd decompositions. The situation, however, is quite different for even decompositions: if $k_j > \sqrt{2n}$, then $k_s/k_j < 2k_s/k_j < \sqrt{2n}$, so the number of even decompositions of n is at most the number of odd decompositions of n. Consequently, if every non-trivial decomposition of n is even, then n has exactly one non-trivial decomposition. Those numbers with this property are precisely the numbers of the form $2^d k$ with k an odd prime greater than 2^{d+1}.

Epilogue

Finding the exact number of decompositions of n can be hard. In general, the formula $\prod_p (e_p + 1)$ is impractical for large n since it essentially calls for the prime factoriza-
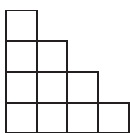
tion of n. What LeVeque obtained in [**2**], among other things, was the average order of the number of decompositions as a function of n. Moreover, he discussed not only the sums of consecutive integers but the sums of arithmetic progressions in general. Readers with a taste for analytic number theory will find his article enjoyable.

Guy gave a very short proof of the theorem in [**1**], then deduced from it a characterization of primes. He gave some rough estimates of the number of decompositions and also remarked that finding this number explicitly is not easy.
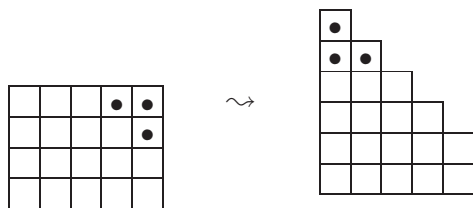
Weil's book [**4**] is merely a collection of exercises for the elementary number theory course given by him at University of Chicago in the summer of 1949. Maxwell Rosenlicht, an assistant of Weil's at that time, was in charge of the "laboratory" section and responsible for most of the exercises. It was a relief to learn from Weil that the challenge of motivating students to work on problems is rather common. The following is part of the foreword taken directly from that book:

> The course consisted of two lectures a week, supplemented by a weekly "laboratory period" where students were given exercises.... The idea was borrowed from the "Praktikum" of German universities. Being alien to the local tradition, it did not work out as well as I had hoped, and student attendance at the problem sessions soon became desultory.

An obvious but crucial point in our proof of the conjecture is that 0 can be expressed as the sum of an odd number of consecutive integers. Let me explain the intuition behind this trick here. The sum of the first n consecutive natural numbers is called the *n-th triangular number* because $1, 2, \ldots, n$ can be arranged to form a triangle (see Figure 1). Now a sum of consecutive numbers can be viewed as the difference of two triangular numbers (or a "trapezoidal number"). Also, a number n, as long as it is not a power of 2, can be represented by an $(n/k) \times k$ rectangle with $k > 1$ an odd factor of n. So the question is: how can you get a trapezoid from such a rectangle? Well, it does not take a big leap of imagination to see that this can be done by cutting off a corner of the rectangle and flipping it over. For example, the diagrams in Figure 2 illustrate how we obtain the decomposition $(2, 3, 4, 5, 6)$ of 20.
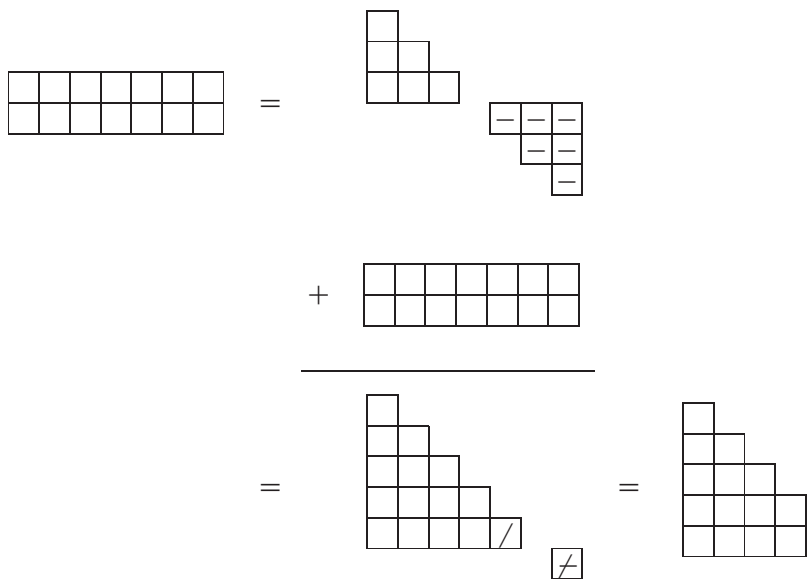

Figure 1. The fourth triangular number is 10.

Figure 2. Transforming a 4×5 rectangle into a trapezoid.

Of course, one has to worry about the case when $n/k < (k - 1)/2$, but this is exactly why the use of negative numbers comes in handy. The diagrams in Figure 3 should be self-explanatory.

Figure 3. The decomposition (2, 3, 4, 5) of 14.

References

1. Robert Guy, Sums of consecutive integers, *Fibonacci Quart*. **20** (1982) 36–38.
2. W.J. LeVeque, On representations as a sum of consecutive integers, *Canad. J. Math*. **2** (1950) 399–405.
3. W.Y. Pong, Length spectra of natural numbers, manuscript.
4. André Weil, *Number Theory for Beginners* (With the collaboration of Maxwell Rosenlicht). Springer-Verlag, 1979.

Art Friedel (friedel@ipfw.edu) of Indiana University Purdue University Fort Wayne found some interesting math on the Kong Size boxes of Kellogg's Raisin Bran.