A Quick Solution of Triangle Counting

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The first law of famous counting problems must be that each solver regards his or her approach as the only truly simple one. Undissuaded, and motivated by the recent reminiscence [1], I offer the following. *Problem*. Find a formula for T_n, the total number of triangles in an equilateral triangle of side n tiled by equilateral triangles of side 1 (see FIGURE 1).

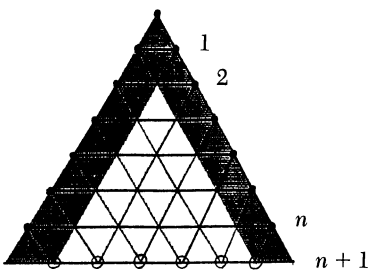

FIGURE 1

A triangle is *new* if it contains a triangle from the bottom row of the diagram. The number of new triangles is $T_{n+1} - T_n$. A triangle is *crusty* if it contains a triangle from the shaded 'crust'. All triangles are either *nablas* (∇) or *deltas* (Δ).

The number of new but not crusty triangles is clearly $T_{n-1} - T_{n-2}$. The new, crusty deltas have their top apices "beaded." There are $2n + 1$ of them. The new, crusty nablas have their bottom apices "circled." There are n of them. Thus

$$T_{n+1} - T_n = T_{n-1} - T_{n-2} + 3n + 1.$$

This is an unremarkable recurrence, which, when supplied with $T_1 = 1$, $T_2 = 5$ and $T_3 = 13$ produces

$$T_n = \frac{(-1)^n + 4n^3 + 10n^2 + 4n - 1}{16}.$$

The characteristic cubic is particularly nice to factor by grouping and the methods of [2, Ch. 5] or [3, Sec. 7.3] determine the coefficients of T_n easily. For computer algebra converts, one call to the *genf* function in MACSYMA produces the answer as shown here.

REFERENCES

1. M. E. Larsen, The eternal triangle—a history of a counting problem, *College Math. J.* 20 (1989) 370–384.
2. M. R. Spiegel, *Calculus of Finite Differences and Difference Equations*, Schaum's Outline Series, McGraw-Hill Book Co., New York, 1971.
3. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley Publishing Co., Reading, PA 1989.