A New (?) Test for Convergence of Series

K. P. S. BHASKARA RAO
Indian Statistical Institute
Bangalore 560 059, India

While teaching a course on real analysis, I wanted to motivate, or, 'discover' the root test for the students. An attempt at this led to the 'discovery' of the following test. With this in mind, along with the proof, I shall include its motivation.

Let Σa_n be a series of strictly positive terms. Define

$$\alpha = \overline{\lim}\,(\log a_n/\log n) \text{ and } \beta = \underline{\lim}(\log a_n/\log n).$$

The test

$$\begin{array}{ll} \text{If } \alpha < -1, & \text{then } \Sigma a_n \text{ converges.} \\ \text{If } \beta > 1, & \text{then } \Sigma a_n \text{ diverges.} \end{array}$$

Proof. We start with a known result about the convergence or divergence of a family of series, depending on a parameter p:

$$\begin{array}{ll} \Sigma \dfrac{1}{n^p} \text{ converges} & \text{if } p > 1 \\ \text{and diverges} & \text{if } 0 \le p \le 1. \end{array}$$

Thus $p = 1$ is the critical value, or, turning point.

We consider applying the comparison test to test the convergence of Σa_n. Let us try to express this in a nice and natural way.

Firstly If there is a $p > 1$ such that $a_n \le (1/n^p)$ for all but finitely many n's, then Σa_n converges.

Rewriting: If there is a $p > 1$ such that $(\log a_n/\log n) \le -p$ for all but finitely many n's, then Σa_n converges.

Rewriting again: If $\overline{\lim}\,(\log a_n/\log n) < -1$, then Σa_n converges.

So, if $\alpha < -1$, then Σa_n converges.

Secondly If there is a $p \le 1$ such that $(\log a_n/\log n) \ge (1/n^p)$ for all but finitely many n's, then Σa_n diverges.

Rewriting: If there is a $p \le 1$ such that $(\log a_n/\log n) \ge -p$ for all but finitely many n's, then Σa_n diverges.

We cannot rewrite this in terms of β. But at least we can say:

If $\underline{\lim}(\log a_n/\log n) > -1$, then there is a $p \le 1$ such that $(\log a_n/\log n) \ge -p$ for all but finitely many n's.

So, if $\beta > -1$, then Σa_n diverges.

Thirdly The above arguments fail for the case $\beta \le -1 \le \alpha$.

Remark 1. A similar analysis with the family $\{\Sigma_n p^n\}_{p>1}$ in place of $\{\Sigma_n (1/n^p)\}_{p>0}$ would give the 'root test' in place of our test. A little care should be exercised in the second part, while proving Σa^n diverges if $\overline{\lim}\, a_n^{1/n} > 1$.

Remark 2. Any family of nonnegative series $\{\Sigma_n f_p(n) : p > 0\}$ for which the convergence or the divergence is known, should, by a similar analysis, lead to another test for convergence.

Remark 3. From the test above it follows that the Dirichlet series $\Sigma_{n \geq 1} a_n n^s$ with $a_n > 0$ for all n

$$\text{converges if} \quad s < -\overline{\lim} \frac{\log a_n}{\log n} - 1$$

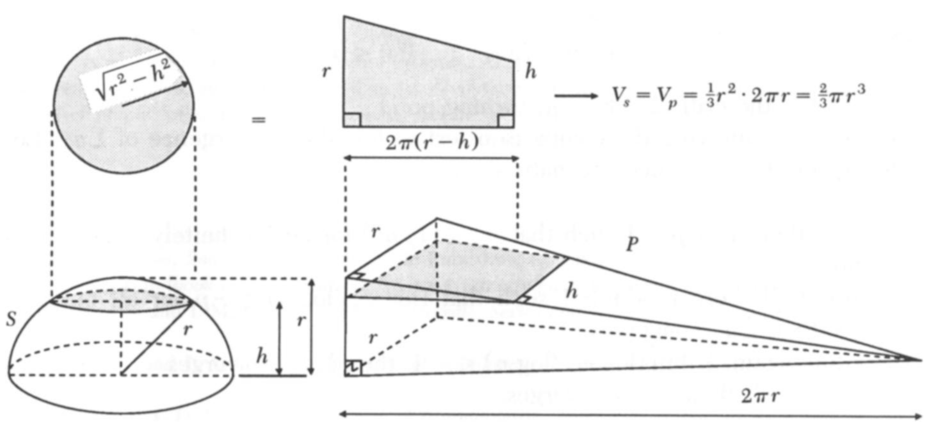
$$\text{and diverges if} \quad s > -\underline{\lim} \frac{\log a_n}{\log n} - 1.$$

Acknowledgement. Thanks to Adam Fieldsteel for pointing out an error in an earlier draft of this note.

REFERENCE

1. Ross, K. A., *Elementary Analysis: The Theory of Calculus*, Springer-Verlag New York, 1986.

Proof without Words: The Volume of a Hemisphere via Cavalieri's Principle*



*Tzu Geng, son of the most celebrated mathematician in ancient China, Tzu Chung Chih, was believed to be the first to develop the principle in the 5th century A.D.

—SIDNEY H. KUNG
JACKSONVILLE UNIVERSITY
JACKSONVILLE, FL 32211