**Theorem.** *There are* $\frac{4}{3}(22^{35}) - \frac{1}{3}$ *distinct bidding auctions.*

*Proof.* There are 35 possible bids other than passes, doubles, and redoubles. Every auction involves exactly one subset of these 35 bids. Once the subset is determined, the order of these bids is determined.

Suppose we are given a subset of size $b$ ($0 < b \leq 35$). The first of these $b$ bids may be preceded by 0, 1, 2, or 3 passes (four possibilities). In between each of the two bids, there are twenty-one possibilities: three in which no one doubles, six in which someone doubles but no one redoubles, and twelve in which someone redoubles. (Recall that one may not double or redouble one's partner.) After the last of the $b$ bids, there are seven possibilities: everyone passes, or either opponent doubles followed by three passes, or either opponent doubles and either of the last bidder's team members redoubles. This provides the total of $4 \times 21^{b-1} \times 7 = \frac{4}{3} \times 21^{b}$ possible auctions, each of which involves precisely $b$ bids. If $b = 0$, then there is one possible auction (everyone passes). Therefore, the total number of possible auctions is

$$\begin{aligned}
1 + &\left\{\binom{35}{1} \times \tfrac{4}{3} \times 21^{1}\right\} + \left\{\binom{35}{2} \times \tfrac{4}{3} \times 21^{2}\right\} + \cdots + \left\{\binom{35}{35} \times \tfrac{4}{3} \times 21^{35}\right\} \\
&= \tfrac{4}{3}(1 + 21)^{35} - \tfrac{1}{3} \\
&= \tfrac{4}{3}(22^{35}) - \tfrac{1}{3} \\
&\approx 1.29 \times 10^{47}.
\end{aligned}$$

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## A General Form of the Arithmetic–Geometric Mean Inequality via the Mean Value Theorem

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In "More Applications of the Mean Value Theorem" [CMJ 16 (November 1985) 397–398], it was shown how the Mean Value Theorem can be used to derive the arithmetic-geometric mean inequality. Here, we show that a similar argument can be used to prove the more general form of this inequality for weighted means.

*Let* $w_1, w_2, \ldots, w_n$ *be positive real numbers whose sum is* 1. *Then, for any positive real numbers* $a_1, a_2, \ldots, a_n$:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}, \tag{1}$$

*with equality holding if and only if* $a_1 = a_2 = \cdots = a_n$.

The arithmetic-geometric mean inequality is the special case of (1), where $w_1 = w_2 = \cdots = w_n = 1/n$. Another important case of (1) is the result that for any positive numbers $p, q, x, y$:

$$\left(\frac{1}{p}\right) x^{p} + \left(\frac{1}{q}\right) y^{q} \geq xy, \tag{2}$$

where $(1/p) + (1/q) = 1$. Inequality (2) is frequently used to derive Hölder's Inequality and also Minkowski's Inequality.

We begin our proof of (1) by applying the MVT to $f(x) = \ln x$ on $[1, b]$ to get $\ln b = (b-1)(1/c)$ for some $c \in (1, b)$. This can be written as

$$\frac{(b-1)}{b} < \ln b < b - 1. \tag{3}$$

Using the left-hand inequality of (3), we have $1 - (1/b) < \ln b$, or

$$(1/b) > 1 + \ln(1/b).$$

Since $b > 1$, it follows that $0 < 1/b < 1$. On the other hand, the right-hand inequality of (3) shows that $b > 1 + \ln b$ for $b > 1$. Thus, we see that for any $x > 0$:

$$x \geq 1 + \ln x. \tag{4}$$

with equality holding if and only if $x = 1$.

To establish (1), let

$$S = w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \quad \text{and} \quad P = a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}.$$

We can show that $S \geq P$ by successively substituting $x = a_i/P$ into (4) and multiplying by $w_i$ for each $i = 1, 2, \ldots, n$. Hence,

$$\begin{aligned} \frac{w_1 a_1}{P} &\geq w_1 + w_1 \ln\left(\frac{a_1}{P}\right) \\ \frac{w_2 a_2}{P} &\geq w_2 + w_2 \ln\left(\frac{a_2}{P}\right) \\ &\vdots \\ \frac{w_n a_n}{P} &\geq w_n + w_n \ln\left(\frac{a_n}{P}\right) \end{aligned} \tag{5}$$

It follows that

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{P} \geq w_1 + w_2 + \cdots + w_n + \ln\left(\frac{a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}}{P^{w_1 + w_2 + \cdots + w_n}}\right).$$

Since $w_1 + w_2 + \cdots + w_n = 1$, we get

$$(S/P) \geq 1 + \ln(P/P),$$

or

$$S \geq P.$$

Equality holds if and only if each $a_i/P = 1$ $(i = 1, 2, \ldots, n)$ in (5); that is, if and only if $a_1 = a_2 = \cdots = a_n$.

*Editor's Note.* Students may find it instructive to establish (1) via Lagrange's method, by maximizing $a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}$ subject to the constraint $w_1 a_1 + w_2 a_2 + \cdots + w_n a_n = S$.

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The figure of 2.2 children per adult female was felt to be in some respects absurd, and a Royal Commission suggested that the middle classes be paid money to increase the average to a rounder and more convenient number.

*Punch* (via M. J. Moroney, *Facts from Figures*.)