Subgroups and the Partitioning Property

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Lagrange's theorem states that the order of a subgroup H of a finite group G divides the order of G. The key step in the standard proof is to show that the left cosets *partition* the group, that is, any two left cosets of the subgroup are either equal or disjoint and the union of all the left cosets is the whole group. In this note, we answer the question: *Are there subsets other than subgroups whose "left cosets" partition G?*

Let X be a subset of elements of a group G. If g is an element of G, the *left translate* gX of X is the set $\{gx: x \in X\}$. A collection $\mathscr{C}$ of nonempty subsets of G is said to *partition* G if any two subsets in $\mathscr{C}$ are either equal or disjoint and their union is G.

THEOREM. *Let X be a subset of a group G. Then the left translates of X partition G if and only if X is a (left or right) coset of a subgroup of G.*

Proof. Suppose first that X is a coset of a subgroup H of G. If X is a right coset and equals Ha, where $a \in G$, then $X = Ha = a(a^{-1}Ha)$. Hence, X is a left coset of the conjugate subgroup $a^{-1}Ha$. Thus, we can suppose that X is a left coset of H and X equals aH. Since $gX = gaH$ and $gH = ga^{-1}aH = ga^{-1}X$, left translates of X are left cosets of H and conversely. We conclude that the left translates of X partition G.

Now suppose that the left translates of X partition G. Let H be the left translate aX containing the identity e of G. Because $gX = ga^{-1}aX = ga^{-1}H$, left translates of H coincide with left translates of X. Therefore, X itself is a left translate of H, the left translates of H partition G, and it remains to show that H is a subgroup of G. Let x and y be elements in H. Because y is in H, the product xy is in xH. Moreover, as e is in H and $x = xe, x$ is in xH. Hence, H and xH are not disjoint and xH equals H. We conclude that xy is in H. Now consider $x^{-1}x$. Since it equals the identity e, it is in H. On the other hand, since x is in $H, x^{-1}x$ is in $x^{-1}H$. Hence, H and $x^{-1}H$ are not disjoint and are equal. We conclude that x^{-1} is in H.

One way in which a partition arises naturally is as the collection of nonempty inverse images $f^{-1}(t)$ of a function $f: E \to S$. In particular, let G be a group of permutations acting on the set S and let s be a fixed element of S. Consider the function $f: G \to S$ defined by $f(\pi) = \pi(s)$, for $\pi \in G$. An inverse image $f^{-1}(t)$ is nonempty if and only if t is in the orbit of s, that is, t equals $\rho(s)$ for some permutation ρ in G. For such an element t and a permutation σ in G,

$$\sigma\left[f^{-1}(t)\right] = \{\sigma\pi: \pi(s) = t\} = \{\tau: \tau(s) = \sigma(t)\} = f^{-1}(\sigma(t))$$

and

$$f^{-1}(\sigma(s)) = f^{-1}(\sigma\rho^{-1}\rho(s)) = \sigma\rho^{-1}\left[f^{-1}(t)\right].$$

Thus, the nonempty inverse images coincide with the left translates of $f^{-1}(t)$. We conclude that the left translates of $f^{-1}(t)$ partition G. By the theorem, they are the cosets of a subgroup. This subgroup is the left translate $f^{-1}(s)$ consisting of the permutations π such that $\pi(s) = s$. We see, then, that the subgroup is the stabilizer $\mathrm{Stab}(s)$ of s. It follows that

$$\text{the number of elements in the orbit of } s = |G|/|\mathrm{Stab}(s)|.$$

A similar argument can be used to show that the kernel of a homomorphism is a subgroup.

When a group G can be depicted geometrically, it is often easy to see whether left translates partition G. Consider the group $\mathbb{C}^\times$ of nonzero complex numbers under multiplication. If $X \subset \mathbb{C}^\times$, then the left translate $re^{i\theta}X$ can be obtained geometrically from X by rotating through the angle θ and dilating (or "stretching") by r. Suppose X is a closed curve (that is, a continuous image of the circle containing at least two points). Then it is intuitively evident that X and a rotation of X through a sufficiently small angle intersect except in the case when X is a circle. (See FIGURE 1.) Using the fact that X is compact, this intuition can be formalized. Thus, except for circles (centered at 0), closed curves are not cosets of subgroups. On the other hand, the left translates of a circle partition $\mathbb{C}^\times$ and are the cosets of the subgroup consisting of the unit circle. Using this fact, we obtain the following result.

THEOREM. *Let H be a subgroup of $\mathbb{C}^\times$ containing a closed curve. Then H is either the whole group $\mathbb{C}^\times$ or the unit circle.*

The condition that the curve be closed is crucial. For example, the unions of open half-lines shown in FIGURE 2 are subgroups since their left translates partition $\mathbb{C}^\times$. However, cosets can partition $\mathbb{C}^\times$ in a nonobvious way. An example is given by the "spiral" subgroup $\{re^{i\log r}: r \in \mathbb{R}\}$.

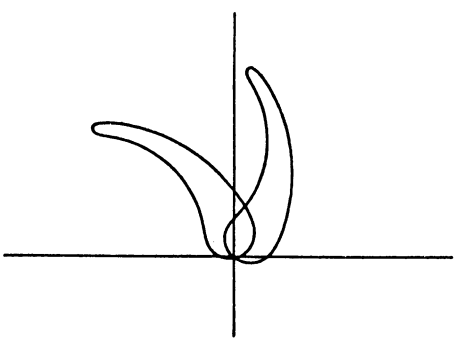

FIGURE 1
A closed curve and its rotation.

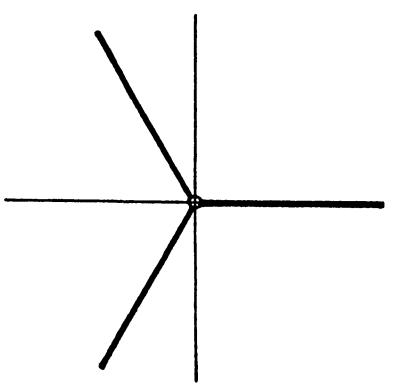

FIGURE 2
A subgroup consisting of three half-lines. The angle between any two lines is 120°.

We end this note with an observation. The theorem holds if we use *right* translates rather than left translates. Thus, it follows that the right translates of a set X partition G if and only if the left translates partition G. It seems relatively difficult (and not elementary) to prove this fact directly without first proving that X is a coset of a subgroup.