CLASSROOM CAPSULES

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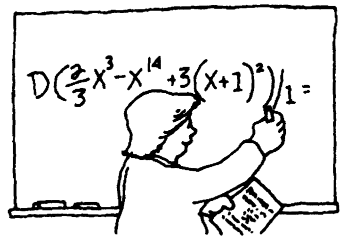

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

Introducing Binary and Ternary Codes via Weighings

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An effective way to introduce binary and ternary codes to students for the first time is to present the following classic puzzle.

> A woman possesses five stones. She claims that, with the use of a two-pan balance, she can match the weight of any rock you hand her if its weight is an integral number of pounds up to 31 pounds. What are the weights of her five stones?

As every integer from 1 to 31 can be written as a sum of some, or all, of the first five powers of two, stones of weight 1, 2, 4, 8, and 16 pounds do the trick. Another way to express this result is that every integer up to and including 31 can be expressed as a five-digit binary number with each digit either 0 or 1.

It is a surprise to learn that the woman can do much better if she relaxes her claim.

> The woman claims that, with the aid of a two-pan balance, she can determine the weight of any rock you give her if it has integral weight no more than 121 pounds. What are the weights of her five stones? (Note: She no longer claims she can match the weight of your rock, only that she can determine its value.)

To solve this puzzle, we move to ternary expansions and consider the option of placing stones on *both* sides of the scale. Every number from 1 up to 121 can be expressed as a ternary number with digits 1, 0, or -1. For example, $64 = 81 - 27 + 9 + 1$, and so 64 is the five-digit number $1\ -1\ 1\ 0\ 1$ in base three. Thus, a rock of 64 pounds placed on one side of the scale along with a 27 pound stone will balance perfectly with the three stones of weights 81, 9, and 1 pounds on the other. In this way, a woman with five stones of weight 1, 3, 9, 27, and 81 pounds can, by trial and error, balance any rock of unknown weight up to 121 pounds with a combination of stones placed on both sides of the scale.

Finally, we mention that the woman with five stones is able to accomplish an even more impressive feat.

> The woman claims that, with the aid of a two-pan balance, she can determine the weight of any rock you give her provided it is of integral weight of at most 242 pounds! What are the weights of her five stones?

The trick here is to double all the weights of the previous puzzle: the woman uses stones of 2, 6, 18, 54, and 162 pounds. As before, she can balance the weight of any rock you hand her with a combination of stones placed on both sides of the balance, provided your rock weighs an even number of pounds. If it doesn't, she can, at the very least, determine that your rock weighs more than one even number and less than the next consecutive even number. From this, she then knows the weight of your rock!

At this point, students may be ready and motivated to consider more challenging related problems, for example:

> How many seemingly identical coins can be tested on a two-pan balance in order to determine whether the coins are equally weighted—and, if not, to identify which is the unique heavy or light coin?

The answer, of course, depends on the number of weighings allowed. Three coins can be tested in two weighings, and up to twelve coins can be tested in three weighings. In general, k ($k \geq 2$) weighings suffice to test any set up to $(3^k - 3)/2$ coins. For a proof with examples, and related references, see [**1**].

Reference

1. Aaron L. Buckman and Frank S. Hawthorne, Coin Weighings Revisited, in *Two-Year College Mathematics Readings* (Warren Page, editor), Mathematical Association of America (1981) 275–281.

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Convergence-Divergence of *p*-Series

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Recently, Khan [**1**] proved the divergence and convergence of the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \tag{1}$$

by rearrangement of the series rather than by the integral test. This capsule illustrates that the p-series divergence-convergence essentially follows from knowlege about the geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots, \tag{2}$$

which converges if $|r| < 1$, and diverges if $|r| \geq 1$.

