Odd-like (Even-like) Functions on (a, b)

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The integral $\int_0^{\pi/2} \cos(\pi \sin^2 \theta)\, d\theta$, which appeared in the famous *Ramanujan's Notebooks* [**1**, p. 56], appears difficult to evaluate and, therefore, served to motivate this note.

For a continuous function $f(x)$ on $[-a, a]$, calculus students know that

$$\int_{-a}^{a} f(x)\, dx = 0$$

when $f(x)$ is an odd function (i.e., $f(-x) = -f(x)$ for all x). Therefore, it would be useful for students to know when a function on (a, b) can be translated to describe an odd function on a corresponding interval $(-c, c)$.

Suppose $f(x)$ is defined on the interval (a, b). Then (see Figure 1), $f(a + b - x) = -f(x)$ for all x in (a, b) is equivalent to $g(x) = f(x + \frac{a+b}{2})$ being an odd function on the interval $(-\frac{b-a}{2}, \frac{b-a}{2})$, since

$$\begin{aligned} g(-x) &= f\left(-x + \frac{a+b}{2}\right) = f\left[a + b - \left(x + \frac{a+b}{2}\right)\right] \\ &= -f\left(x + \frac{a+b}{2}\right) = -g(x) \end{aligned}$$

for all $x \in (-\frac{b-a}{2}, \frac{b-a}{2})$.

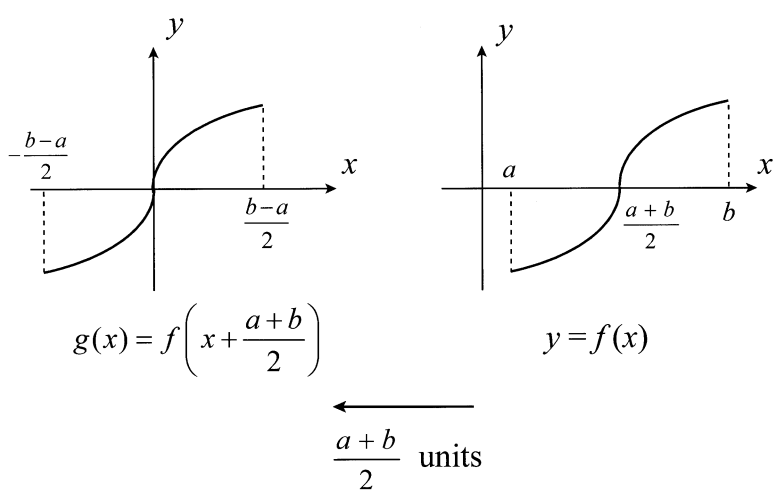

Figure 1.

Definition. A function $f(x)$ is *odd-like* on the interval (a, b) if $f(a + b - x) = -f(x)$ for all x in (a, b). (Thus, $f(x)$ is odd if and only if it is odd-like on $(-a, a)$ for all $a > 0$. And $f(x)$ is odd on $(-a, a)$ if and only if it is odd-like on $(-a, a)$.)

If $f(x)$ is an integrable, odd-like function on (a, b), then the translated function $g(x) = f(x + \frac{a+b}{2})$ is an odd function on $(-\frac{b-a}{2}, \frac{b-a}{2})$ and so

$$\int_a^b f(x)\,dx = \int_{-\frac{(b-a)}{2}}^{\frac{b-a}{2}} g(x)\,dx = 0.$$

This also can be verified directly by letting $x = a + b - u$ and observing that

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-u)\,du = \int_a^b -f(u)\,du = -\int_a^b f(x)\,dx.$$

Examples. Each function below is odd-like on the indicated interval.

(i) $\cos nx$ on $(0, \pi)$ with n odd, and $\sin nx$ on $(0, \pi)$ with n even.

(ii) $\frac{\sin(mx)}{\sin(nx)}$ on $(0, \pi)$ for positive integers m and n, when $m + n$ is odd.

(iii) $\cos(n\pi \sin^2 x)$ and $\cos(n\pi \cos^2 x)$ on $(0, \pi/2)$ when n is odd.

(iv) $\sin(n\pi \sin^2 x)$ and $\sin(n\pi \cos^2 x)$ on $(0, \pi/2)$ when n is even.

The odd-like property for (i) and (ii) can be verified directly. For $\cos(n\pi \sin^2 x)$ in (iii), observe that

$$\begin{aligned}\cos\left(n\pi \sin^2\left(\frac{\pi}{2} - x\right)\right) &= \cos(n\pi \cos^2 x) = \cos(n\pi(1 - \sin^2 x)) \\ &= \cos(n\pi - n\pi \sin^2 x) = -\cos(n\pi \sin^2 x).\end{aligned}$$

Verification of the remaining functions follows similarly. Each of the above functions has $\int_a^b f(x)\,dx = 0$ on the indicated interval (a, b). Thus,

$$\int_0^{\pi/2} \cos(\pi \sin^2 \theta)\,d\theta = 0$$

is the value of the integral that motivated this note.

For any given function $f(x)$ on (a, b), we can construct an odd-like function on (a, b). Figure 2 illustrates why $F(x) = f(a + b - x) - f(x)$ is an odd-like function on (a, b).

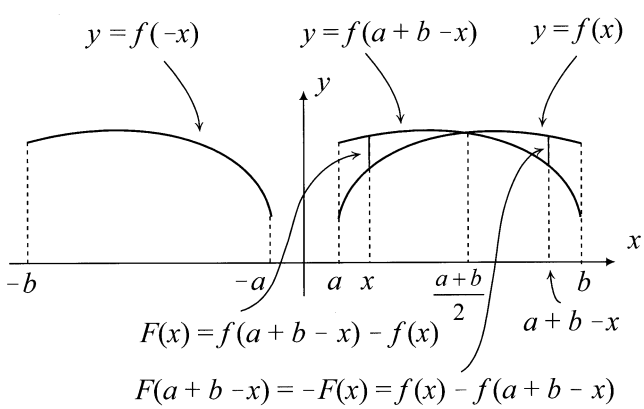

Figure 2.

Example. The function $\ln(\frac{\pi-x}{x}) = \ln(\pi - x) - \ln x$ is odd-like on $(\frac{\pi}{4}, \frac{3\pi}{4})$ since $\ln(\frac{\pi-(\pi-x)}{\pi-x}) = -\ln(\frac{\pi-x}{x})$. Thus, $\int_{\pi/4}^{3\pi/4} \ln(\frac{\pi-x}{x})\,dx = 0$.

Proposition. If $f(x)$ is an integrable function and $f(x) \neq -f(a+b-x)$ for any x in (a, b), then

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}.$$

Proof. The function $\frac{f(x)-f(a+b-x)}{f(x)+f(a+b-x)}$ is an odd-like function on (a, b), and so the first equality holds. (This also follows from the substitution $x = a+b-u$.) And since $\frac{f(x)}{f(x)+f(a+b-x)} = 1 - \frac{f(a+b-x)}{f(x)+f(a+b-x)}$, it follows that $2\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = b-a$. ■

Examples.

(i) $\displaystyle\int_\pi^{3\pi} \frac{\ln x}{\ln x + \ln(4\pi - x)}\,dx = \pi.$

(ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^r x}{\sin^r x + \cos^r x}\,dx = \frac{\pi}{12}$ for any real number r.

(iii) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x^r}{x^r + (\pi - x)^r}\,dx = \frac{\pi}{6}$ for any real number r.

The counterpart of odd-like functions are even-like functions defined as follows.

Definition. A function $f(x)$ is *even-like* on the interval (a, b) if $f(a+b-x) = f(x)$ for all x in (a, b). (Thus, a function $f(x)$ is even if and only if it is even-like on $(-a, a)$ for all $a > 0$. And $f(x)$ is even on $(-a, a)$ if and only if it is even-like on $(-a, a)$.)

The reasoning for odd-like functions can be used to show that $f(x)$ is even-like on (a, b) if and only if $g(x) = f(x + \frac{a+b}{2})$ is an even function on $(-\frac{b-a}{2}, \frac{b-a}{2})$. If $f(x)$ is an integrable, even-like function on (a, b), then $g(x)$ is even on $(-\frac{b-a}{2}, \frac{b-a}{2})$ and so

$$\int_a^b f(x)\,dx = \int_{-\frac{(b-a)}{2}}^{\frac{b-a}{2}} g(x)\,dx = 2\int_0^{\frac{b-a}{2}} g(x)\,dx = 2\int_{\frac{a+b}{2}}^b f(x)\,dx.$$

Note also that for every function $f(x)$, the function $F(x) = f(a+b-x) + f(x)$ is even-like on (a, b).

Examples.

(i) $\cos nx$ is even-like on $(0, \pi)$ when n is even, and $\sin nx$ is even-like on $(0, \pi)$ when n is odd.

(ii) $\ln(x(4\pi - x)) = \ln x + \ln(4\pi - x)$ is even-like on the interval $(\pi, 3\pi)$.

Students may find it instructive to prove the following.

Exercises.

(1) If $f(x)$ is differentiable and odd-like on (a, b), then $f'(x)$ is even-like on (a, b).

(2) If $f(x)$ is differentiable and even-like on (a, b), then $f'(x)$ is odd-like on (a, b).

(3) If $f(x)$ is odd-like (even-like) on (a, b) and n is a positive even number, then the nth derivative $f^{(n)}(x)$ is even-like (odd-like) on (a, b).

(4) If $f(x)$ is a continuous, odd-like function on (a, b), then any antiderivative of $f(x)$ is even-like on (a, b).

(5) If $f(x)$ is a continuous, even-like function on (a, b), then there is exactly one antiderivative of $f(x)$ that is odd-like on (a, b).

(6) The set of all odd-like functions on (a, b) is a vector space, as is the set of all even-like functions on (a, b).

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Reference

1. Bruce C. Berndt, *Ramanujan's Notebooks*, Part II, Springer-Verlag, New York, 1985.

Keyboard Inequalities

Monte Zerger (mjzerger@adams.edu) observes that the F, $F^\sharp$, and G keys on the piano demonstrate mathematical truth. The Pythagoreans ("all is number") found that the ratio of the frequencies in a perfect fifth is 3 to 2 and in a fourth is 4 to 3. We do not hear these when we strike piano keys because, to avoid having to have twelve pianos around the house, each tuned to a different key, we have adopted well-tempered tuning where the ratio of frequencies for any half-tone is $2^{1/12}$ to 1. So, in any octave we have

Interval	Well-tempered Tuning	Pythagorean Tuning	Mean of the numbers 1 and 2
$F : C$	$2^{5/12} = 1.335\ldots$	$4/3 = 1.333\ldots$	Harmonic: $4/3$
$F^\sharp : C$	$2^{6/12} = 1.414\ldots$	$729/512 = 1.424\ldots$	Geometric: $\sqrt{2}$
$G : C$	$2^{7/12} = 1.498\ldots$	$3/2 = 1.5$	Arithmetic: $3/2$

The three keys are telling us that

$$\text{Harmonic mean} \leq \text{Geometric mean} \leq \text{Arithmetic mean}.$$