Acknowledgment. The author wishes to thank his colleagues S. Kumar and Elmer Young for their helpful comments.

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2. J. Stewart, *Calculus: Concepts and Contexts*, 2nd ed., Brooks/Cole, 2001.

Combinatorial Proofs via Flagpole Arrangements

Duane DeTemple (detemple@wsu.edu), Washington State University, Pullman, WA 99164-3113

In their recent book [**1**], Arthur Benjamin and Jennifer Quinn derive a wide array of algebraic and number theoretic identities with clever applications of elementary counting principles. As they say, "While [counting is] not necessarily the simplest approach, it offers another method to gain understanding of mathematical truths. To a combinatorialist, this kind of proof is the only right one." A sampling of the combinatorial approach was presented in [**2**], which also issued a challenge to the reader to find a combinatorial proof of the identity

$$\sum_{k=1}^{n} k^2 = \frac{1}{4}\binom{2n+2}{3}^2.$$

In this note, I will take on this challenge by considering arrangements of flagpoles and guy wires. This scheme has several nice features. Primarily, it is easy to visualize and therefore suggests some useful correspondences between arrangements that would remain hidden in more abstract set correspondences. Moreover, it is quite versatile, so that a number of identities can be obtained by varying the rules of the valid arrangements. In particular, it is easy to obtain a new combinatorial proof of the sum of cubes identity shown in [**2**]. In the concluding example, a flagpole arrangement problem is devised that gives a purely combinatorial derivation of the identity

$$\sum_{k=1}^{n-1} k^2(n-k) = \frac{1}{3}\binom{n}{2}\binom{n+1}{2}.$$

The proof of the following identity serves as a warm up to the flagpole arrangement method.

Identity A.

$$\sum_{k=0}^{n-1} \binom{k}{q}\binom{n-1-k}{r} = \binom{n}{q+r+1}$$

Question: Consider n blocks, each of which can either support a flagpole or anchor a single guy wire. The pole is to be supported by q blue guy wires attached to distinct blocks to the left of the pole, together with r red guy wires attached to distinct blocks to the right of the pole. In how many ways can the pole and its supporting guy wires be arranged?

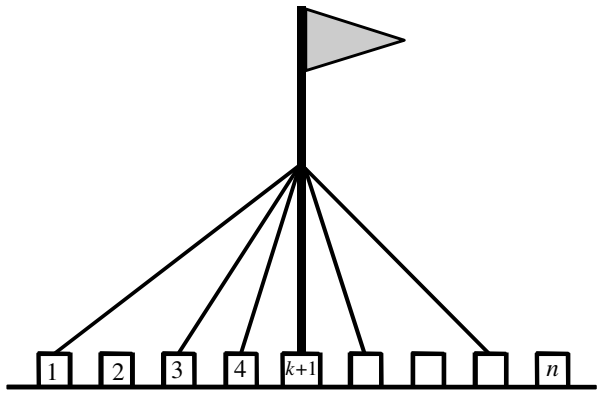

Figure 1.

Answer 1: Suppose the flagpole is on block $k+1$, where $0 \le k \le n-1$. Then there are $\binom{k}{q}$ ways to choose the blocks for the q blue guy wires to the left of the pole and $\binom{n-1-k}{r}$ ways to choose the blocks to the right of the pole for the red guy wires. Therefore there are

$$\sum_{k=0}^{n-1} \binom{k}{q} \binom{n-1-k}{r}$$

arrangements.

Answer 2: Simply choose $q+r+1$ of the n blocks, using the first q blocks to anchor blue guy wires, the next block for the pole, and the last r blocks to anchor the red guy wires. This gives $\binom{n}{q+r+1}$ arrangements, and proves the identity. ■

Identity B.

$$\sum_{k=1}^{n-1} k^2 = \frac{1}{4}\binom{2n}{3}$$

Question: A flagpole is placed on one of the n blocks and anchored with a blue and a green guy wire attached to blocks (or perhaps to the same block) to the left of the pole. In how many ways can the pole and the two guy wires be erected?

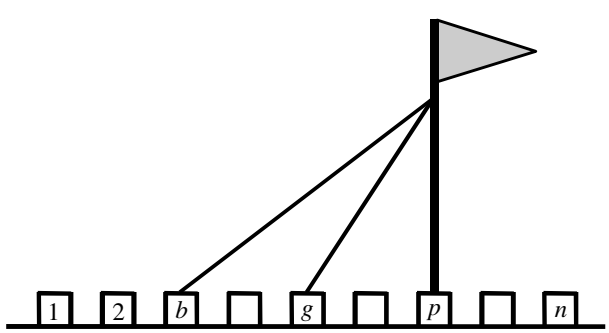

Figure 2.

Answer 1: As before, if the flagpole is placed on block $k+1$, then there are k^2 ways to place the blue and green guy wires on the k blocks to the left of the pole. This gives $\sum_{k=1}^{n-1} k^2$ arrangements. We also notice that each arrangement corresponds to a unique element of the set $S = \{(b, g, p) \mid 1 \le b, g < p \le n\}$. Indeed, b and g give the block numbers that anchor the blue and green guy wires, respectively, and p gives the larger block number on which the pole rests. Therefore, $|S| = \sum_{k=1}^{n-1} k^2$.

Answer 2: Now let

$$T = \left\{(i, j, k) \mid 1 \le i < j < k \le n', \text{ where } i, j, k \in \{1, 1', 2, 2', \ldots, n, n'\}\right\},$$

so that

$$|T| = \binom{2n}{3}.$$

There is a one-to-four mapping of S onto T:

$$(b, g, p) \to \begin{cases} (b, g, p), (b', g, p), (b, g', p), (b', g', p), & \text{if } b < g \\ (g, b, p'), (g, b', p'), (g', b, p'), (g', b', p'), & \text{if } b > g \\ (b, b', p), (b, b', p'), (b, p, p'), (b', p, p'), & \text{if } b = g. \end{cases}$$

Thus, $|S| = \frac{1}{4}|T|$ and the identity is proved. ■

Identity C.

$$\sum_{k=1}^{n-1} k^3 = \binom{n}{2}^2$$

Question: A flagpole is placed on one of n blocks. It is anchored with a blue, green, and red guy wire attached to blocks all to the left of the pole, where a block can be used to anchor any number of wires. In how many ways can the pole and the three guy wires be erected?

Answer 1: Reasoning as before, there are $\sum_{k=1}^{n-1} k^3$ arrangements. We can also let b, g, and r denote the block numbers that anchor the blue, green, and red guy wires, respectively, and let p denote the block on which the pole rests. Therefore, each arrangement corresponds to a unique member of the set $S = \{(b, g, r, p) \mid 1 \le b, g, r < p \le n\}$ and so $|S| = \sum_{k=1}^{n-1} k^3$.

Answer 2: Let $T = \{(\{h, i\}, \{j, k\}) \mid 1 \le h < i \le n, 1 \le j < k \le n\}$. That is, T is the set of ordered pairs of two-element subsets of $\{1, 2, \ldots, n\}$, so $|T| = \binom{n}{2}^2$. The elements of S and T can be matched with the following one-to-one correspondence:

$$(b, g, r, p) \to \begin{cases} \big(\{b, g\}, \{r, p\}\big), & \text{if } b < g \\ \big(\{r, p\}, \{g, b\}\big), & \text{if } b > g \\ \big(\{b, p\}, \{r, p\}\big), & \text{if } b = g \end{cases}$$

The identity follows since $|S| = |T|$. T can also be described directly in terms of flagpole arrangements: First, the flagpole is located at the largest element p in either of the two sets. If p occurs only in the second set, then the other element

in the set is the site number of the red guy wire, and the first set gives the locations of the blue and green guy wires in that order from left to right. If p occurs only in the first set, the other element of the set is the site of the red guy wire and the second pair give the locations of the green and blue guy wires in that order. Finally, if p occurs in both sets, the other element of the first set is the common site of both the blue and green wires, and the other element of the second set is the site of the red wire. ■

The concluding example combines the ideas used to derive the identities above.

Identity D.

$$\sum_{k=1}^{n-1} k^2(n-k) = \frac{1}{3}\binom{n}{2}\binom{n+1}{2}$$

Question: A flagpole is placed on one of $(n+1)$ blocks in a row. It is anchored with a blue and a green guy wire to blocks (perhaps the same block) to the left of the pole, and with a red guy wire attached to a block to the right of the pole. In how many ways can the pole and the three guy wires be erected?

Answer 1: If the pole is on block $k+1$, then there are then k^2 ways to attach the blue and green guy wires on blocks to the left of the pole and $n-k$ blocks to the right of the pole on which to attach the red guy wire. This gives the answer $\sum_{k=1}^{n-1} k^2(n-k)$. Each arrangement corresponds to a unique quadruple in the set $S = \{(b, g, p, r) \mid 1 \le b, g < p < r \le n+1\}$, so $|S| = \sum_{k=1}^{n-1} k^2(n-k)$.

Answer 2: Let $T =$

$$\big\{\big(\{h, i\}, \{j, k\}\big) \mid 1 \le h < i \le n+1, 1 \le j < k \le n+1 \text{ and } h \ne j \text{ and } i \ne j\big\}.$$

That is, T is the set of ordered pairs of two-element sets from $\{1, 2, \ldots, n+1\}$ where neither element of the first set is equal to the smaller number of the second set. To count the number of elements of T, we first pick the right-hand set of the ordered pair in $\binom{n+1}{2}$ ways, and then pick the left-hand set in the $\binom{n}{2}$ ways that avoid the smaller member of the set already selected. Thus $|T| = \binom{n}{2}\binom{n+1}{2}$. There is a one-to-three mapping of S onto T:

$$(b, g, p, r) \to \begin{cases} \big(\{b, g\}, \{p, r\}\big), \big(\{b, p\}, \{g, r\}\big), \big(\{b, r\}, \{g, p\}\big), & \text{if } b < g \\ \big(\{p, r\}, \{g, b\}\big), \big(\{b, r\}, \{g, p\}\big), \big(\{b, p\}, \{g, r\}\big), & \text{if } b > g \\ \big(\{b, r\}, \{p, r\}\big), \big(\{p, r\}, \{b, r\}\big), \big(\{p, r\}, \{b, p\}\big), & \text{if } b = g \end{cases}$$

and the identity follows since $|S| = \frac{1}{3}|T|$. ■

A "proof without words" derivation of Identity D, due to the elusive C. G. Wastun, is given in [**3**, p. 91]. The expression on the left side of the identity also answers this question from combinatorial geometry:

"How many lattice squares are found in a square array of n^2 lattice points?"

To see why the answer is given by $\sum_{k=1}^{n-1} k^2(n-k)$, refer to Figure 3.

For any k, $1 \le k \le n$, consider any lattice square such as $QRST$ with $n-k+1$ lattice points along each of its vertical and horizontal sides. The upper left vertex Q must be in the upper left $k \times k$ sublattice of the $n \times n$ lattice, so there are k^2 such squares

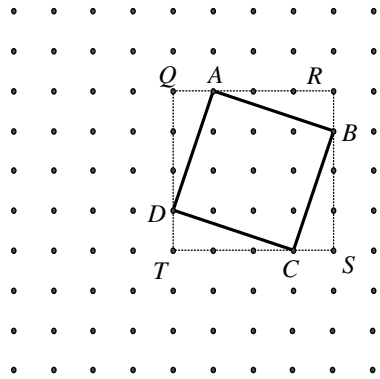

Figure 3.

with sides aligned to the axes of the grid. Each of these squares circumscribes $n-k$ lattice squares, since this is the number of choices of the upper leftmost vertex A of a square $ABCD$ that is inscribed in $QRST$. Altogether, we see there are $\sum_{k=1}^{n-1} k^2(n-k)$ lattice squares in the $n \times n$ lattice.

I'll leave to the reader the challenge of providing combinatorial reasoning that shows the number of squares in an $n \times n$ lattice is also given by $\frac{1}{3}\binom{n}{2}\binom{n+1}{2}$. Neither C. G. Wastun nor I has yet been successful in this effort, so we welcome your ideas.

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Periodic Points for the Tent Function

David Sprows (david.sprows@villanova.edu), Department of Mathematical Sciences, Villanova University, 800 Lancaster Ave., Villanova, PA 19085-1699

The tent function is a piece-wise linear function from the unit interval to itself obtained by letting $f(x) = 2x$ for $0 \le x < 1/2$ and $f(x) = 2 - 2x$ for $1/2 \le x \le 1$. This function is a rich source of examples that can be used to illustrate various concepts in iteration theory (see [**1**]). In particular, the tent function has periodic points of all possible periods. In this note we will investigate some of the properties of these periodic points.

The material in this note is especially suitable as a classroom unit designed to introduce students to some of the basic ideas of iteration theory. This classroom unit can be presented at just about any level since there are essentially no prerequisites other than a familiarity with how base 2 expressions work. It can also serve as a supplement to