it is clearly not congruent to $\triangle A'B'C'$, and so we have shown that there must be two different triangles with the same perimeters and the same areas.

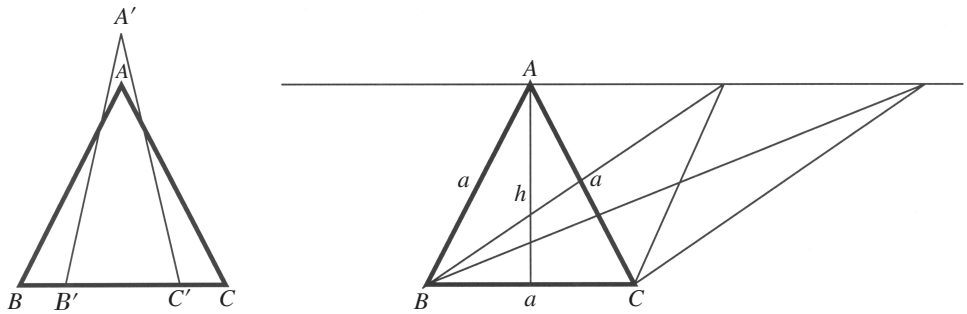

Figure 4.

I should like to note in closing that our isosceles triangle argument, in which the triangles all have the same perimeter and the area varies, can be modified to isosceles triangles for which the area is constant and the perimeter varies. The details are left to the reader.

Note. The author wishes to thank the editors for suggesting some of the ideas presented here.

References

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Can You Paint a Can of Paint?

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The paradox of "Gabriel's horn" is a favorite topic of many calculus teachers. (See, for example, [**1**, p. 402].) In this note, I offer two ways to resolve the paradox.

The "horn" is the surface S gotten by revolving the curve $y = 1/x$ for $x \geq 1$ about the x-axis. The surface has infinite area, but the volume of the 3-dimensional region R inside it is finite. Consider how surprising that is: a finite volume of paint is sufficient to fill R, and then every point of S will be in contact with paint—yet no quantity of paint, however large, will be enough to cover S with paint!

Certainly S has infinite area while R has finite volume—the calculations that lead to those results are not in doubt—so any resolution of the paradox requires us to question the interpretation of the calculations. I will argue that the paradox arises when we make erroneous assumptions about the relationships between area and paint, and between area and volume. Once those relationships are clarified, the apparent contradiction dissolves.

First resolution. Implicit in the paradox is the following assumption.

(A) *The area of a surface is a measure of the amount of paint needed to paint the surface.*

But paint exists in 3-space. If we are planning to paint a room, we might ask the paint-store clerk for enough paint to cover 1000 square feet, but we wouldn't ask for 1000 square feet of paint. Let's replace (A), then, with the following assumption, which captures more accurately the way real paint behaves.

(B) *There is a minimum thickness t such that, if we cover a surface with a coat of paint having thickness at least t, then we will be unable to see the surface behind the paint.*

"Painting a surface" thus means using a finite volume of paint to cover the surface with a coat of thickness at least t. Under this assumption, it is impossible to paint our surface S since the coat of paint would essentially have to be an infinitely long solid tube, the volume of which would be infinite. (The tube would be hollow, but the hollow space would get extremely narrow.)

More precisely, suppose that the paint covering S occupies the hollow 3-dimensional tube obtained by revolving the region between $y = 1/x$ and $y = 1/x + h(x)$ for $x \geq 1$ about the x-axis. Then by our assumption, there's a constant c such that $h(x) \geq c$ for all $x \geq 1$. (A subtle point: $c \neq t$ except where the graph of h is horizontal.) So the volume of paint is

$$\pi \int_1^\infty \left\{ \left[\frac{1}{x} + h(x) \right]^2 - \frac{1}{x^2} \right\} dx > \pi \int_1^\infty \left(\frac{2c}{x} + c^2 \right) dx = \infty,$$

and so S is unpaintable, as claimed.

But now when we fill the space inside S with paint, there is no contradiction. Even though every point of S is in contact with paint, we do not consider S to have been painted, since the thickness of paint inside S approaches 0. For large x, we don't even see the paint inside S—the space filled by the paint is too narrow to be visible.

But that is not the end of this chapter of the story. Under Assumption (B), *no* unbounded surface of revolution can be painted, even if it has finite area! (Nice examples of functions that revolve to such surfaces are $y = 1/x^2$ and $y = e^{-x}$.) The argument for this unpaintability is simply the one given above, but with $1/x$ replaced by the given function.

That conclusion may seem startling: isn't the volume of paint required simply t times the surface area? The answer is "yes" if the surface is flat, as most walls are, but "no" if the surface is curved, as are all surfaces of revolution. To take a simple example, let's consider a right circular cylinder of radius r and height h. Then the lateral surface area A is $2\pi rh$. But paint to a thickness t forms a shell that has volume $V = \pi ht(2r + t) = At(1 + t/(2r))$. Hence, $V \approx At$ only when t is small when compared to r. This makes sense since then we can estimate V well by cutting the lateral surface along a line parallel to the cylinder's axis, unrolling the surface, and treating the coat of paint as a rectangular prism; not so if t is not small compared to r. Thus the formula At provides a rather poor estimate of the volume of paint even for *bounded* surfaces that are highly curved. (See [**2**] for a more detailed discussion of the relationship between volume and surface area, along with a geometrically appealing derivation of the standard integral formula for the surface area obtained by revolving the graph of f on $[a, b]$ about the x-axis.)

Second resolution. I formulated Assumption (B) to reflect our ordinary experience that several coats of paint may be needed in order to cover a wall. But you may object that your paint is no ordinary paint—yours is a special, *mathematical* paint that's an opaque, continuous fluid. Any thickness of such paint, however small, is sufficient to hide the wall. Your paint, then, satisfies the following assumption:

(C) *Any thickness of paint will cover a surface.*

"Painting a surface" now means covering it with paint of positive thickness, the thickness possibly varying with the point being covered. Under this assumption, a sufficiently smooth surface of revolution will *always* be paintable (even if the volume inside is infinite). To prove that, we let $g(x)$ be a continuous, positive, decreasing function that approaches 0 as $x \to \infty$, and we consider the hollow tube obtained by revolving the region between the curves $g(x)$ and $g(x)(1 + 1/x^2)$ for $x \geq 1$ about the x-axis. Then we see from elementary calculus that this tube has finite volume, and so to paint the surface of revolution of $g(x)$ we simply fill our tube with paint. There is no paradox. We normally think of the boundary surface of a solid as being "smaller" than the solid itself, so there is nothing surprising about the case where our container has infinite volume yet can be covered with a finite volume of paint.

Mathematical models. Is there any reason to prefer one of our assumptions over the other? Or is there perhaps some other assumption more compelling than either of ours? Why do we need *any* assumption?

My answer to the third question is that the Gabriel's horn paradox is essentially an exercise in mathematical modeling; in making a model we first make assumptions. After all, the paradox offends our sensibilities not because of any internal logical inconsistency but because the conclusion contradicts our everyday experience. If we regard Assumptions (A), (B), and (C) as bases for models of the process of painting a surface, then we can view the paradox as simply pointing out that a certain prediction that follows from Assumption (A) does not agree with experience. We therefore discard (A) in the hope of replacing it by another, more realistic one.

Of the two candidates (B) and (C) offered here, (B) seems to me the more realistic. But both are unrealistic in that they assume that paint is continuous. Since real paint is composed of molecules, and since molecules have small but finite size, we cannot in reality fill Gabriel's horn with paint—and again there is no paradox.

References

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A Paradoxical Paint Pail

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We are all familiar with Gabriel's horn from calculus [**1**], [**2**]. (See also the preceding Capsule.) It is the object obtained by rotating the graph of $y = 1/x$ for $x \geq 1$