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Another Proof for the *p*-series Test

Yang Hansheng (hsyang921@yahoo.com.cn), Southwest University of Science and Technology, Mianyang, China 621002, and Bin Lu (binlu@csus.edu), California State University Sacramento, Sacramento, CA 95819

It is well known that the p-series is $1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$ converges for $p > 1$ and diverges for $p \leq 1$. In standard calculus textbooks (such as [**3**] and [**4**]), this is usually shown using the integral test. In this note, we provide an alternative proof of the convergence of the p-series without using the integral test. In fact, our proof is an extension of the nice result given by Cohen and Knight [**2**].

We begin by giving the following estimate for the partial sum of a p-series:

Lemma. *Let $s_n(p)$ be the nth partial sum of the p-series $\sum_{k=1}^{\infty} 1/k^p$.*

(a) *For $p > 0$,*

$$1 - \frac{1}{2^p} + \frac{2}{2^p} s_n(p) < s_{2n}(p) < 1 + \frac{2}{2^p} s_n(p),$$

(b) *For $p < 0$,*

$$1 + \frac{2}{2^p} s_n(p) < s_{2n}(p) < 1 - \frac{1}{2^p} + \frac{2}{2^p} s_n(p).$$

Proof. As $s_n(p)$ is the nth partial sum,

$$\begin{aligned} s_{2n}(p) &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2n)^p} \\ &= 1 + \left[\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right] + \left[\frac{1}{3^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n-1)^p}\right]. \end{aligned}$$

For $p > 0$,

$$s_{2n}(p) > 1 + \frac{1}{2^p} s_n(p) + \left[\frac{1}{4^p} + \frac{1}{6^p} + \cdots + \frac{1}{(2n)^p}\right].$$

Thus,

$$s_{2n}(p) > 1 + \frac{1}{2^p} s_n(p) - \frac{1}{2^p} + \frac{1}{2^p} s_n(p) = 1 - \frac{1}{2^p} + \frac{2}{2^p} s_n(p).$$

Also,

$$s_{2n}(p) < 1 + \frac{1}{2^p}s_n(p) + \left[\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right] = \frac{2}{2^p}s_n(p) + 1.$$

This proves (1). We can prove (2) in a similar manner. ■

From these estimates, we have the following test for the p-series:

Theorem. *The p-series is divergent when $p \leq 1$, and in this case,*

$$\lim_{n\to\infty} \frac{s_{2n}(p)}{s_n(p)} = \frac{2}{2^p}. \tag{1}$$

The p-series is convergent for $p > 1$, and in this case,

$$\frac{2^p - 1}{2^p - 2} \leq \lim_{n\to\infty} s_n(p) \leq \frac{2^p}{2^p - 2}. \tag{2}$$

Proof. When $p < 0$, the p-series is divergent since the general term does not converge to 0. So we consider $0 < p \leq 1$. Assume that the p-series is convergent, that is, $\lim_{n\to\infty} s_n(p) = s(p)$. From the lemma we obtain the following inequality by letting $n \to \infty$:

$$1 - \frac{1}{2^p} + \frac{2}{2^p}s(p) = \frac{2^p - 1}{2^p} + \frac{2}{2^p}s(p) \leq s(p),$$

and from this inequality we have

$$0 < \frac{2^p - 1}{2^p} \leq \frac{2^p - 2}{2^p}s(p) \leq 0,$$

which is a contradiction. Thus the p-series is divergent when $p \leq 1$. We obtain

$$\lim_{n\to\infty} \frac{s_{2n}(p)}{s_n(p)} = \frac{2}{2^p}$$

by dividing both inequalities of the lemma by $s_n(p)$ and letting $n \to \infty$. This proves (1).

Now let $p > 1$. From the inequality of the first part of the lemma, we have

$$s_n(p) < s_{2n}(p) < 1 + \frac{2}{2^p}s_n(p),$$

and then $0 < (1 - (2/2^p))s_n(p) < 1$. Hence, $s_n(p) < 2^p/(2^p - 2)$ for all n, so the sequence $\{s_n(p)\}$ is bounded. Furthermore, it is increasing, so the limit $\lim_{n\to\infty} s_n(p)$ exists, and hence the p-series is convergent for $p > 1$. The inequality is obtained by letting $n \to \infty$ in the first part of the lemma. ■

Remarks. (1) From the theorem, for $p > 1$, we obtain an estimate for the sum of a p-series. For example, when $p = 2$, we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.65.$$

The theorem gives the estimate $1.5 \leq \lim_{n\to\infty} s_n(p) \leq 2$.

(2) The theorem also provides a way of calculating some interesting limits related to the p-series. For example, consider the p-series $\sum_{k=1}^{\infty} 1/k^p$ with $p = 1/3$. It is divergent and $\lim_{n\to\infty} s_n(p) = \infty$. Then from the first part of the theorem, we can calculate the limit:

$$\lim_{n\to\infty} \frac{1 + \frac{1}{\sqrt[3]{2}} + \cdots + \frac{1}{\sqrt[3]{n}} + \cdots + \frac{1}{\sqrt[3]{2n}}}{1 + \frac{1}{\sqrt[3]{2}} + \cdots + \frac{1}{\sqrt[3]{n}}} = \sqrt[3]{4}.$$

Similarly for $p = 1$,

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{2n}}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} = 1.$$

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Taylor Series—A Matter of Life or Death

Mathematics can even be a matter of life or death. During the Russian revolution, the mathematical physicist Igor Tamm was seized by anti-communist vigilantes at a village near Odessa where he had gone to barter for food. They suspected he was an anti-Ukrainian communist agitator and dragged him off to their leader.

Asked what he did for a living, he said that he was a mathematician. The sceptical gang-leader began to finger the bullets and grenades slung around his neck. "All right," he said, "calculate the error when the Taylor series approximation of a function is truncated after n terms. Do this and you will go free; fail and you will be shot." Tamm slowly calculated the answer in the dust with his quivering finger. When he had finished, the bandit cast his eye over the answer and waved him on his way.

Tamm won the 1958 Nobel prize for Physics but he never did discover the identity of the unusual bandit leader. But he found a sure way to concentrate his students' minds on the practical importance of Mathematics!

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