$$= \frac{a^2 \sinh^{-1}(x/a)}{2} + \frac{x\sqrt{a^2+x^2}}{2}$$

$$\int \sqrt{a^2+x^2}dx = \frac{a^2 \sinh^{-1}(x/a)}{2} + \frac{x\sqrt{a^2+x^2}}{2} + C.$$

Reference

1. George B. Thomas and Ross L. Finney, *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley, Reading, MA, 1996, p. 528.

Area Relations on the Skewed Chessboard

Larry Hoehn (hoehnl@apsu.edu) Austin Peay State University, Clarksville, TN 37044

To the most casual observer it is obvious that the area of the red (white) squares is equal to the area of the black squares on an ordinary chessboard or checkerboard. But does this still hold on a skewed chessboard such as in Figure 1? The purpose of this article is to answer this and other related questions.

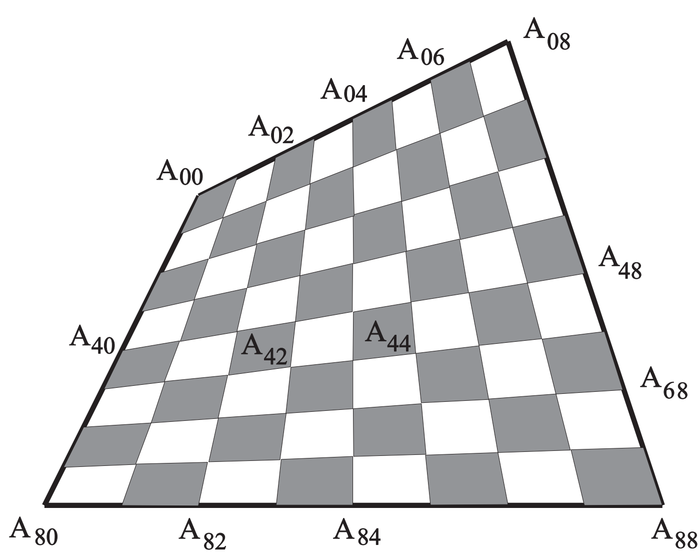

Figure 1.

First we consider more precisely what is meant by a skewed chessboard. Our *skewed chessboard* is any convex quadrilateral in which each side is divided into eight congruent segments whose corresponding endpoints are joined by cross-segments to form sixty-four non-overlapping quadrilaterals.

Since the sides of quadrilateral $A_{00}A_{08}A_{88}A_{80}$ are divided into eight congruent parts, it is natural to wonder whether the cross-segments, such as $A_{40}A_{48}$, are also divided into eight congruent parts. To prove that this is the case, we make repeated use of a result attributed to Pierre Varignon (1654–1722).

Varignon's Theorem. If B, D, F, H are the consecutive midpoints of the sides of quadrilateral $ACEG$, then $BDFH$ is a parallelogram.

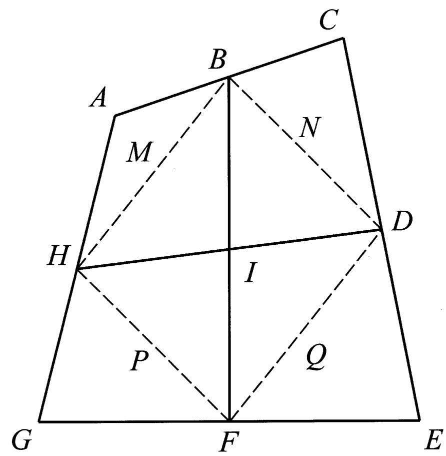

Figure 2.

Proof. ([**1**, p. 53]). By the midsegment, or midline theorem, the line through the midpoints B and D of $\triangle ACE$ is parallel to and one-half the third side, AE. Similarly HF is parallel to and one-half AE for $\triangle AGE$. Therefore, BD and HF are both parallel and congruent, which implies that $BDFH$ is a parallelogram.

As a corollary of Varignon's Theorem, we note that HD and BF bisect each other. This is the actual result needed.

Theorem 1. *Each cross-segment of an 8 by 8 skewed chessboard is divided into eight congruent segments.*

Proof. By repeated use of the corollary to Varignon's Theorem, we can show that each lattice point A_{ij} (except the original perimeter points of the chessboard) of Figure 1 is the midpoint of some quadrilateral's cross-segments. For example, A_{44} is the midpoint of cross-segments $A_{40}A_{48}$ and $A_{04}A_{84}$ of quadrilateral $A_{00}A_{08}A_{88}A_{80}$. Then A_{42} is the midpoint of cross-segments $A_{40}A_{44}$ and $A_{02}A_{82}$ of quadrilateral $A_{00}A_{04}A_{84}A_{80}$. Likewise, A_{41} is the midpoint of cross-segments $A_{40}A_{42}$ and $A_{01}A_{81}$ of quadrilateral $A_{00}A_{02}A_{82}A_{80}$, and so on. Thus, A_{41}, A_{42}, $A_{43}, \ldots, A_{47}$ divide $A_{40}A_{48}$ into eight congruent segments. This process can be repeated to show that each cross-segment, whether "horizontal" or "vertical," is divided into eight congruent segments.

Next we concentrate on results involving a few blocks (i.e., "squares" of the chessboard) rather than the entire chessboard. For convenience we use "$\triangle ABC$" as both the symbol for the triangle and also the area of the triangle. Additionally we will also use a single capital letter to denote the area of a quadrilateral.

Theorem 2. *If three adjacent blocks of a skewed chessboard adjoin in a single row (or column), then the area of the middle one is the arithmetic mean of the areas of the other two.*

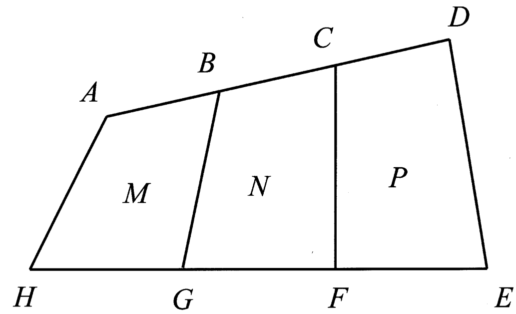

Figure 3.

Proof. Let B, C, F, and G be the respective trisection points of sides AD and EH of quadrilateral $ADEH$ in Figure 3. Additionally, let M, N, and P be the areas of the smaller quadrilaterals formed by the trisection points. We wish to show that $N = (1/2)(M + P)$. It is well known that if two triangles have the same height, then their areas are proportional to their bases. In Figure 3, for example, $\triangle ABG = \triangle CBG$ and $\triangle AHG = (1/3)\triangle AHE$. Therefore,

$$\begin{aligned}
M + P &= (\triangle AHG + \triangle ABG) + (\triangle EDC + \triangle EFC) \\
&= \tfrac{1}{3}\triangle AHE + \triangle CBG + \tfrac{1}{3}\triangle EDA + \triangle GFC \\
&= \tfrac{1}{3}(\triangle AHE + \triangle EDA) + (\triangle CBG + \triangle GFC) \\
&= \tfrac{1}{3}(M + N + P) + N = \tfrac{1}{3}(M + P) + \tfrac{4}{3}N
\end{aligned}$$

Hence, $M + P = 2N$ which completes the proof.

This result can easily be extended to more than 3 adjacent blocks. For example, if M, N, P, Q are four adjacent blocks in a single row (or column), then $M + Q = N + P$ since $M + P = 2N$ and $N + Q = 2P$ by Theorem 2. Similarly, if five blocks M, N, P, Q, R are in a single row, then $M + R = N + Q = 2P$.

Theorem 3. *If four adjacent blocks of a skewed chessboard adjoin so that all share a common vertex, then the sum of the areas of two "diagonal" blocks is equal to the sum of the areas of the other two blocks.*

Proof. In Figure 2, the vertex I is shared by the four blocks of quadrilateral $ACEG$. We must show that $M + Q = P + N$. Since the diagonals of parallelogram $BDFH$ bisect each other,

$$\triangle HIB = \triangle BID = \triangle DIF = \triangle FIH.$$

Since B and H are the midpoints of two sides of $\triangle ACG$, and D and F are the midpoints of two sides of $\triangle CEG$,

$$\triangle ABH + \triangle DEF = \left(\tfrac{1}{4}\right)\triangle ACG + \left(\tfrac{1}{4}\right)\triangle CEG = \left(\tfrac{1}{4}\right)(\text{quad } ACEG).$$

In the same manner, $\triangle HGF + \triangle BCD = (1/4)(\text{quad } ACEG)$. By combining these equalities we obtain

$$
\begin{aligned}
M+Q &= (\triangle ABH+\triangle BIH)+(\triangle DEF+\triangle FID)\\
&= (\triangle ABH+\triangle DEF)+(\triangle BIH+\triangle FID)\\
&= \left(\tfrac{1}{4}\right)(\text{quad } ACEG)+(\triangle BID+\triangle FIH)\\
&= (\triangle HGF+\triangle BCD)+(\triangle BID+\triangle FIH)\\
&= (\triangle HGF+\triangle FIH)+(\triangle BID+\triangle BCD)\\
&= P+N.
\end{aligned}
$$

We note that Theorem 3 is a known result and appears in [**2**].

Combining Theorems 1, 2, and 3, we can prove additional results by adding and simplifying several equations. Thus (Figure 4), if we have a 3 by 3 skewed quadrilateral where each side is trisected, we obtain:

(1) $A+F=D+C$
(2) $A+I=2E=G+C$
(3) $B+H=D+F$
(4) $B+F+G=D+H+C$, and
(5) $E=\frac{1}{9}(A+B+C+D+E+F+G+H+I)$.

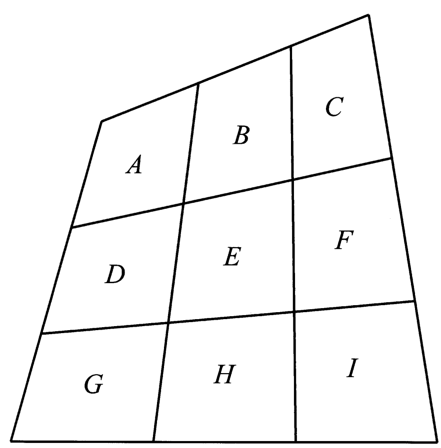

Figure 4.

In [**3**], Greenberg gave an arduous, but clever, proof of (5). Since this note was inspired by (5), we now prove it as follows.

$$
\begin{aligned}
&A+B+C+D+E+F+G+H+I\\
&\quad = (A+G)+(C+I)+(B+H)+(D+F)+E\\
&\quad = 2D+2F+2E+2E+E\\
&\quad = 2(D+F)+5E\\
&\quad = 2(2E)+5E\\
&\quad = 9E.
\end{aligned}
$$

Therefore $E=\frac{1}{9}$ times the area of the quadrilateral.

Using Theorems 1, 2, and 3, the reader can obtain many additional area relations on n by n skewed chessboards. Finally, we return to the original question. The cross-segments in bold print of Figure 1 divide the skewed chessboard $A_{00}A_{08}A_{88}A_{80}$ into sixteen 2 by 2 blocks. By Theorem 3, the sum of the areas of the black blocks is equal to the sum of the areas of the white blocks for each of the sixteen 2 by 2 blocks. Therefore, the total area of the white blocks equals the total area of the black blocks. Do comparable relationships hold for cubes in a skewed 3-dimensional chessboard?

References

1. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, 1967.
2. Botand Köszegi, Solution to Problem #2 of the Arany David Competition 1987, *Crux Mathematicorum* **16** (1990) 264.
3. Benjamin Greenberg, That area problem, *Mathematics Teacher* **64** (1971) 79–80.

On the Monotonicity of $\left(1+\frac{1}{n}\right)^n$ and $\left(1+\frac{1}{n}\right)^{n+1}$

Peter R. Mercer (mercerpr@math.buffalostate.edu), SUNY College at Buffalo, NY 14222.

Since the function $f(t) = 1/t$ is decreasing on $(0, +\infty)$, for $0 < a < b$ we have

$$f(b)[b-a] \leq \int_a^b f(t)\,dt \leq f(a)[b-a].$$

For $a = n$ and $b = n + 1$, this reduces to

$$\frac{1}{n+1} \leq \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n}. \tag{1}$$

The inequalities (1) imply (upon multiplication by n) that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$. Although $\{(1+\frac{1}{n})^n\}$ is an increasing sequence and $\{(1+\frac{1}{n})^{n+1}\}$ is a decreasing sequence, this cannot be proved by (1) alone; one must use, for example, the Mean Value Theorem, or the Binomial Theorem, or the Arithmetic-Geometric Mean Inequality [**4, 3, 2**]. Below we refine inequalities (1) to prove these two results, and we get a little bit more.

For any convex function $F(t)$,

$$F\left(\frac{a+b}{2}\right)[b-a] \leq \int_a^b F(t)\,dt \leq \frac{F(a)+F(b)}{2}[b-a]. \tag{2}$$

(The right-hand side is the area of the trapezoid circumscribed at the endpoints, and the left-hand side is the area of the trapezoid inscribed at the midpoint. This is known as Hadamard's Inequality [**1**].) For $F(t) = 1/t$, with $a = n$ and $b = n + 1$, the inequalities (2) become

$$\frac{2}{2n+1} < \log\left(1+\frac{1}{n}\right) < \frac{2n+1}{2n(n+1)}. \tag{3}$$

