



*La Géométrie* was a difficult work to read, as Descartes left much to the reader to work out for himself. He wrote “I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.”

The popularity of Descartes’ *La Géométrie* is due largely to **Franz Van Schooten** (1615–1660), a Dutch professor of mathematics who translated Descartes’ work into Latin and wrote a commentary on it filling in many gaps. Van Schooten was particularly interested in conic section drawers and wrote a **treatise** on them. He devised several instruments for drawing conic curves, as shown in Figure 2. Like Descartes’ instruments, these generally consisted of a collection of straight rods hinged together in some way.

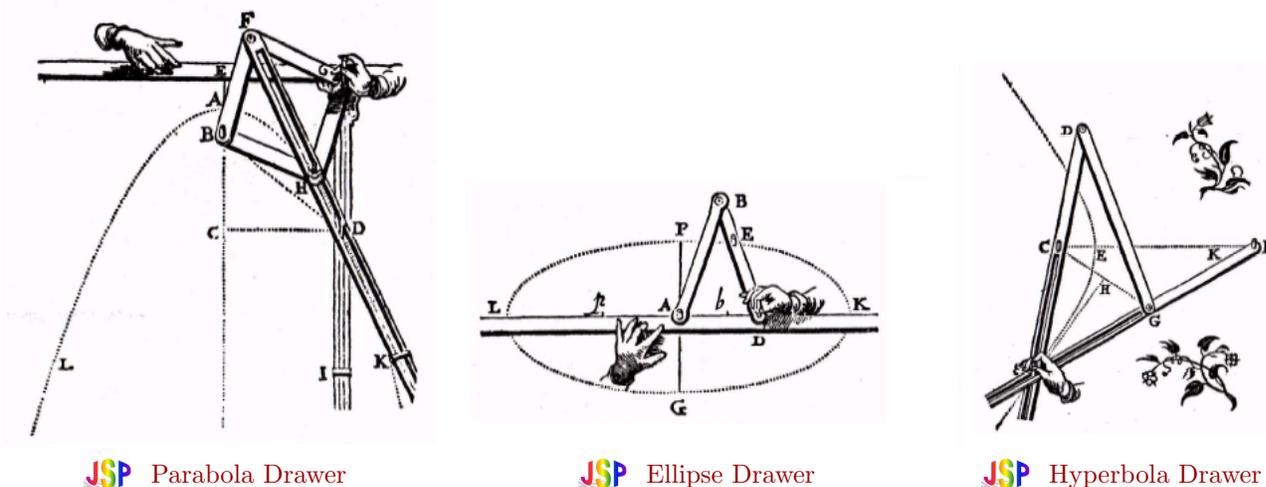


Figure 2: Franz Van Schooten’s Conic Section Drawers

Today’s graphing calculators easily generate graphs of curves by using an algebraic equation to generate the coordinates of many points on the curve and then plotting them. Many, many points must be plotted in order to obtain an accurate graph, so this would not have been a very practical method before the age of computers. In contrast, the ancients drew curves by constructing specialized instruments or “curve-drawers”. In the same way that ruler and compass can be used to draw straight lines and circles in a continuous movement, the curve drawers devised by Van Schooten and others were used to draw other curves of interest, particularly the conic sections, namely parabolas, ellipses, and hyperbolas.

In the exercises in this first module, several ellipse drawers from the history of mathematics are explored. These include the familiar string construction, as well as an ellipse drawer due to **Proclus** (411–485), and another one attributed to **Archimedes** (287 BCE - 212 BCE).

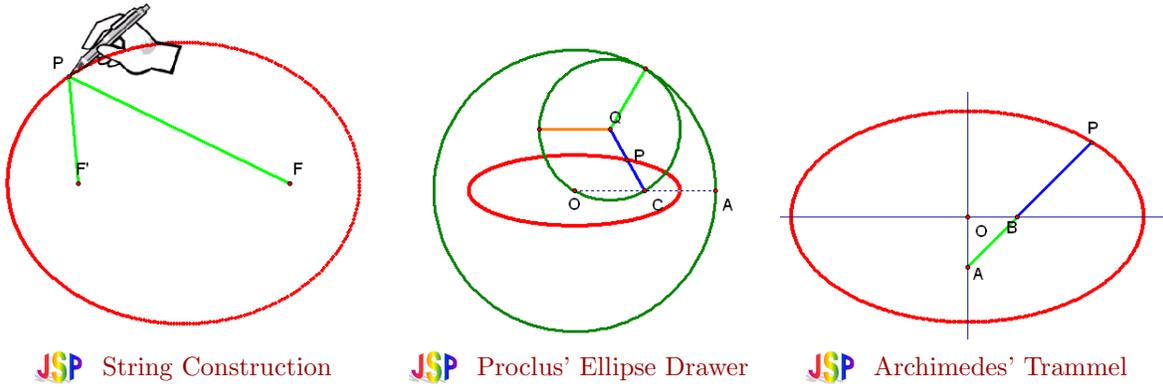


Figure 3: Ellipse Drawers

- [Module 1 Exercises](#)

- [Geometer's Sketchpad file needed for Module 1](#)



## References

- [1 ] Carl B. Boyer, *Historical stages in the definition of curves*, National Mathematics Magazine, Vol. 19, No. 6 (March 1945), pp. 294–310.
- [2 ] Jan Van Maanen, *Seventeenth Century Instruments for Drawing Conic Sections*, The Mathematical Gazette, Vol. 76, No. 476 (July 1992), pp. 222-230.

## Module 2: Tangent Lines Then and Now

As long as mathematicians have been studying curves, they have been interested in the construction of their tangent lines. **Apollonius**, who lived over two thousand years ago and studied the conic curves extensively, found geometric methods for constructing tangent lines to parabolas, ellipses, and hyperbolas. In fact, Propositions 33 and 34 of Book I of Apollonius' great work *The Conics* give recipes for constructing tangents to these curves.

Apollonius' recipe for constructing tangent lines to parabolas is as follows:

**Proposition I-33** *Let  $P$  be a point on the parabola with vertex  $E$ , with  $PD$  perpendicular to the axis of symmetry. If  $A$  is on the axis of symmetry so that  $AE = ED$ , then  $AP$  will be tangent to the parabola at  $P$ .*

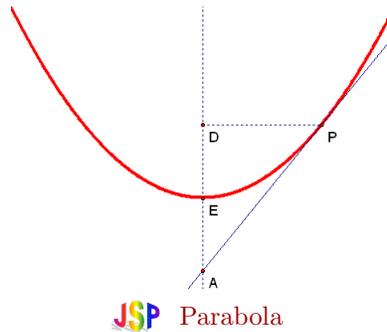


Figure 4: Apollonius' Recipe for constructing tangent lines to parabolas

Apollonius' recipe for construction tangent lines to ellipses and hyperbolas is as follows:

**Proposition I-34** *Let  $P$  be a point on an ellipse or hyperbola,  $PB$  the perpendicular from the point to the main axis. Let  $G$  and  $H$  be the intersections of the axis with the curve and choose  $A$  on the axis so that  $\frac{|AH|}{|AG|} = \frac{|BH|}{|BG|}$ . Then  $AP$  will be tangent to the curve at  $P$ .*

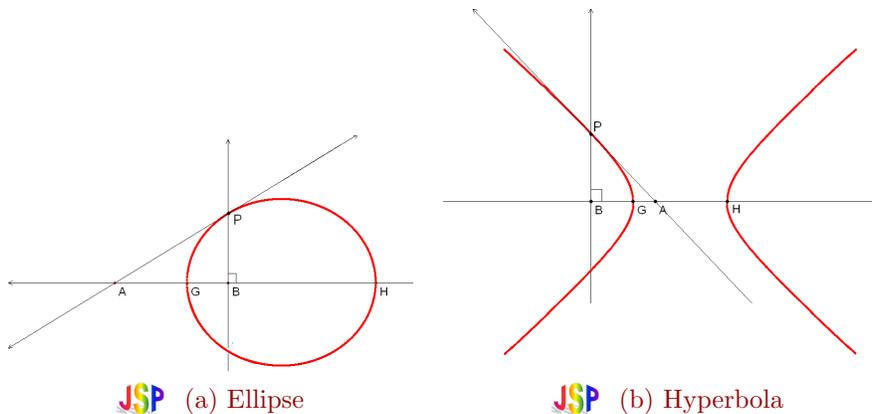


Figure 5: Apollonius' Recipe for constructing tangent lines to ellipses and hyperbolas

Apollonius' recipes for constructing tangent lines to conic curves are easy to implement, but his methods were limited in that they could only be applied to a handful of curves. Of course, in the days of antiquity, there weren't that many curves that people were interested in studying, and so there wasn't that much of a need for a general method.

By the beginning of the seventeenth century, the collection of known curves had grown. Curves were often described as the path of a moving point (e.g. a circle can be viewed as the path traced out by a point on the outer edge of a spinning wheel). Indeed, the curve drawing instruments that were used to draw curves, some of which were discussed in Module 1, underscored this notion of a curve as a path of a moving

point. It was **Gilles Personne de Roberval** (1602–1675) who exploited this definition of a curve by viewing a tangent line as the direction in which the point is moving at a particular instant.

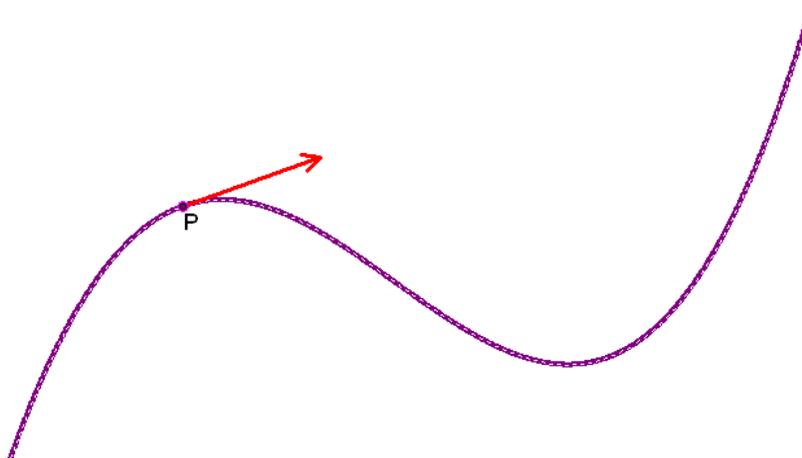


Figure 6: Tangent as direction of motion

Consider, for example, a parabola, which, recalling Van Schooten's parabola drawer, can be defined as the path of point  $D$  below as  $G$  moves along  $AC$ .

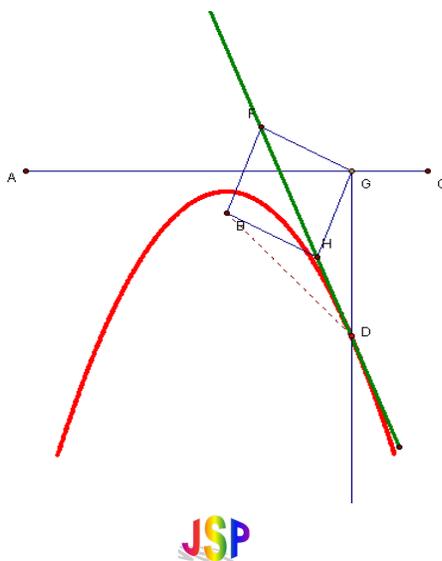


Figure 7: Van Schooten's Parabola Drawer

As  $G$  moves along  $AC$ , the point  $D$  as being pulled simultaneously toward the point  $B$  and toward the horizontal line  $AC$ . Since  $|GD| = |BD|$ , the two forces are equal (if  $D$  moves a little toward  $B$ , it must move by the same amount toward the line  $AB$ ). By the parallelogram law, the direction of motion of  $D$  should be the angle bisector of  $\angle BDG$ . This angle bisector is actually the diagonal  $FH$  of the rhombus  $FGHB$ .

Both Apollonius and Roberval used geometrical methods to construct tangent lines. One of the first algebraic method is due to **René Descartes**, one of the inventors of analytic geometry. It was well known that a tangent line to a circle is always perpendicular to the radius of the circle. So Descartes' idea was to first find a circle tangent to the given curve. Then the tangent to that circle is the sought after tangent to the curve.

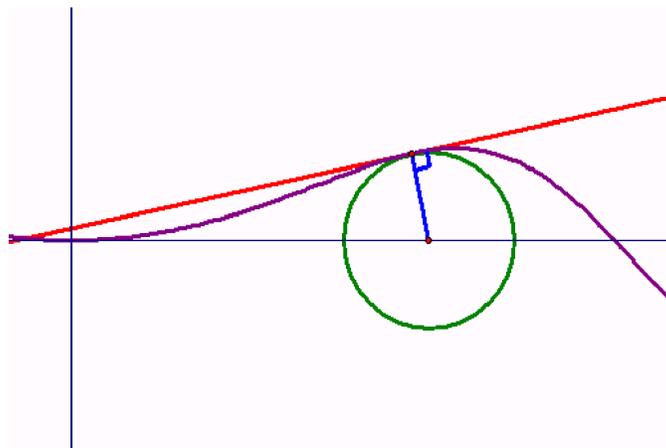


Figure 8: Descartes' Method: Tangent of curve is perpendicular to radius of tangent circle

It was **Pierre de Fermat** who came up with essentially the method used today to construct tangent lines, by viewing the tangent as the limit of secant lines through the given point  $A$  and a nearby point  $B$  on the curve, where  $B$  approaches and eventually becomes  $A$ .

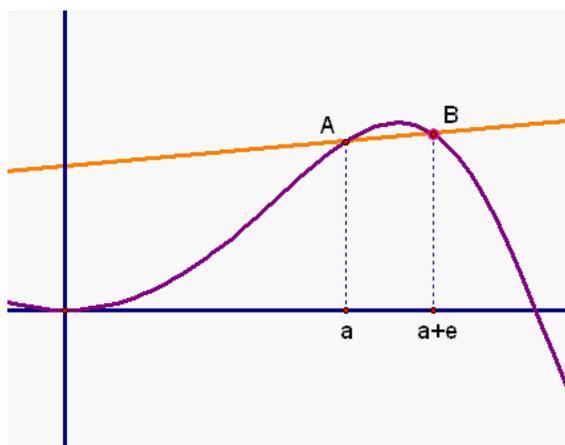


Figure 9: Fermat's Method of Constructing Tangent Lines

Fermat's method, although it worked in practice, was not without controversy. If  $A = (a, f(a))$  is a point on the curve  $y = f(x)$ , and  $B = (a + e, f(a + e))$  is a nearby point, the slope of the line  $AB$  is perfectly well-defined as long as  $A$  and  $B$  are distinct (i.e.  $e \neq 0$ ), but how do we define the slope when  $e = 0$  so that  $B = A$ ? This seeming inconsistency invited much criticism, the most famous coming from the Irish philosopher **Bishop George Berkeley** (1685–1753), in a pamphlet he published in 1734 called *The Analyst*. Here Bishop Berkeley ridiculed the increments  $e$ , calling them the “ghosts of departed quantities”. In fact, Descartes developed his method discussed above as a way to get around these difficulties. You can see below how, as  $B$  approaches  $A$ , the circle with center  $O$  on the  $x$  axis that goes through  $A$  and  $B$  is well-defined, even when  $A = B$ . However the line through  $A$  and  $B$  cannot be well-defined when  $A = B$ .

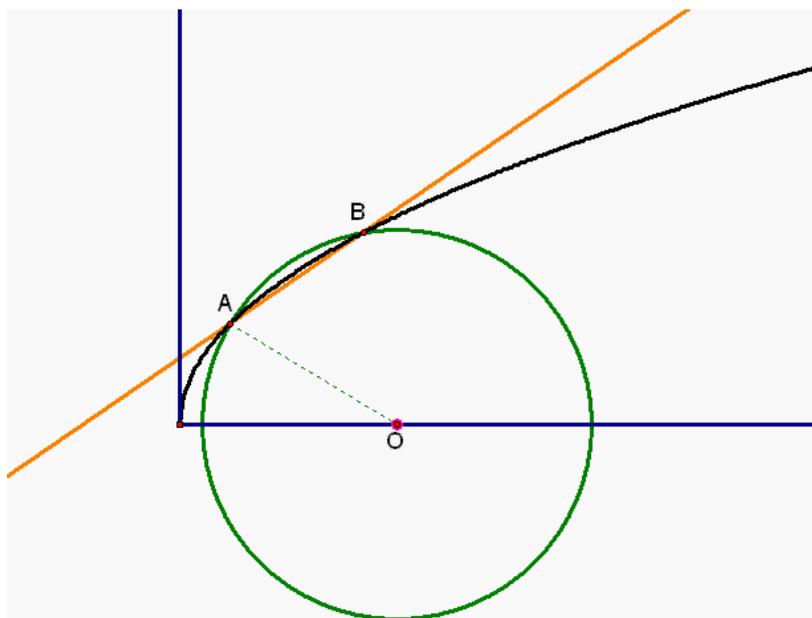


Figure 10: Descartes' Method Compared to Fermat's Method

Unfortunately, Descartes' method did not work very well except for some simple curves, because of the burdensome algebra involved in finding the unique circle with center on the  $x$  axis that intersects the curve only once at  $A$ .

In spite of the problems with the logical foundations of the subject, nobody could dispute that the techniques worked. And so Fermat is credited with the invention of the differential calculus since he developed this method in 1629 and used it successfully to compute subtangents of a large collection of curves. It wasn't until the nineteenth century that the controversies and problems were satisfactorily addressed by the French mathematician **Augustin-Louis Cauchy** (1789–1857), who gave a precise definition of derivative in terms of a new concept called *limit*. This led to the definition that we use today.

- [Module 2 Exercises](#)

- [Geometer's Sketchpad file needed for Module 2](#)



## References

- [1 ] J. L. Coolidge, *The Story of Tangents*, The American Mathematical Monthly, Vol. 58, No. 7. (1951), pp. 449-462.
- [2 ] Paul R. Wolfson, *The Crooked Made Straight: Roberval and Newton on Tangents*, The American Mathematical Monthly, Vol. 108, No. 3 (2001), pp. 206-216.

### Module 3: Optimization Problems Then and Now

Perhaps the most important application of the differential calculus is the solution of optimization problems, where one wants to find the value of a variable that maximizes or minimizes a certain quantity. In this module we look at the history of some of these problems and their solutions.

- Heron's "Shortest Distance" Problem

One of the first non-trivial optimization problems was solved by **Heron of Alexandria**, who lived about 10–75 A.D. Heron's "shortest distance" problem is as follows: Given two points  $A$  and  $B$  on one side of a straight line, to find the point  $C$  on the line such that  $|AC| + |CB|$  is as small as possible. Thus in the picture below, one possible point  $C$  is shown, as well as the length of the path  $A \rightarrow C \rightarrow B$ .

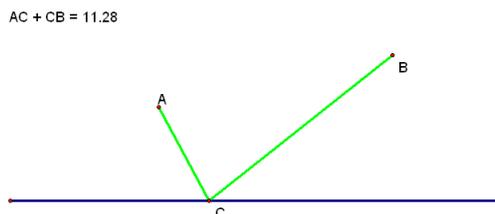


Figure 11: A possible path from  $A$  to  $B$

Click [JSP here](#) to explore the solution to Heron's problem. You can drag  $C$  to determine the approximate length of the shortest path attainable.

To solve this problem mathematically, Heron noticed that if  $B$  is reflected across the line, to say  $B'$ , then for any point  $C$  on the line,  $|CB| = |CB'|$ , and hence minimizing  $|AC| + |CB|$  is equivalent to minimizing  $|AC| + |CB'|$ . But clearly since the shortest path from  $A$  to  $B'$  is a straight line, the point  $C$  that minimizes  $|AC| + |CB'|$  should be the point of intersection of the line with the line segment  $AB'$ . Any other path, such as  $A \rightarrow C' \rightarrow B'$ , will clearly be longer.

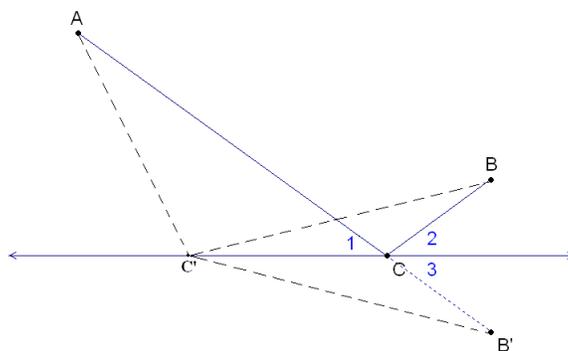


Figure 12: Heron's solution of the "Shortest Distance" problem

Note that  $\angle 2 = \angle 3$  by construction, and  $\angle 1 = \angle 3$  since these are vertical angles. Therefore,  $\angle 1 = \angle 2$ . This is the **equal angle law of reflection**. It was Euclid who, over three hundred years earlier, had noted the now well-known reflection law of light:

**Equal Angle Law of Reflection** If a beam of light is sent toward a mirror, then the angle of incidence equals the angle of reflection.

Heron appears to have been the first to observe that the reflection law implies that **light always takes the shortest path**.

As a matter of fact, the mirror in the equal angle law of reflection need not be flat. We may replace the line in Heron's problem by any concave curve  $S$  (a curve is concave if it lies entirely on one side of any tangent line). In this case, the angles are measured with respect to the tangent line, and the same argument used by Heron shows that if  $C$  is such that the angle of incidence equals the angle of reflection, then  $|AC| + |BC|$  is minimized.

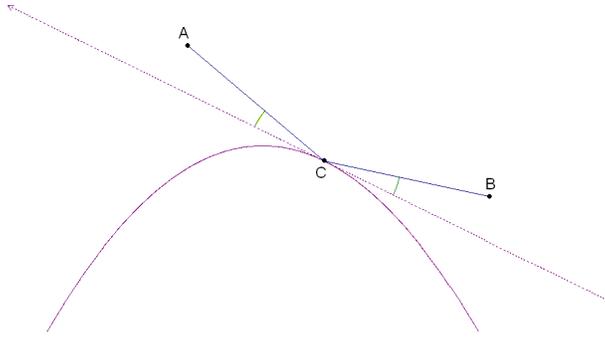


Figure 13: Law of reflection for any concave curve

An interesting application of the law of reflection arises in the case of a light beam sent toward a parabolic mirror, where the light beam is parallel to the axis of the parabola. A parabolic mirror is one whose surface is generated by rotating a parabola about its axis. Suppose the parabola has focus  $F$  and directrix  $\mathcal{L}$ , and that the light beam  $\overrightarrow{GA}$  hits the parabola at  $A$ . Recall from Roberval's construction of tangent lines to parabolas that the tangent line at  $A$  bisects the angle  $\angle FAB$ . Hence  $\angle 1 = \angle 2$ . Since  $\angle 2$  and  $\angle 3$  are vertical angles,  $\angle 2 = \angle 3$ . Hence  $\angle 1 = \angle 3$ . So by the Equal Angle Law of Reflection, the light beam will be reflected in the direction  $\overrightarrow{AF}$ . This will be true for any point  $A$  on the parabola.

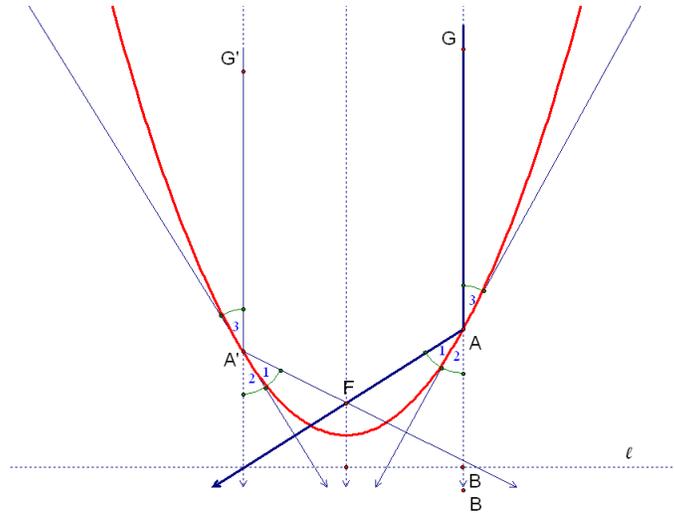


Figure 14: Light rays directed toward a parabolic mirror parallel to the parabola's axis are all reflected toward the focus  $F$  of the parabola

There is a famous story about Archimedes in which he is said to have used a parabolic mirror to defeat the Roman General Marcellus. Supposedly, he tilted the mirror toward the sun in such a way that all

the sun's rays when reflected off the mirror went through the focal point. The heat generated at that point caused a fire to ignite and destroy the entire Roman fleet.



Figure 15: Wall painting from the Stanzino delle Matematiche in the Galleria degli Uffizi (Florence, Italy). Painted by Giulio Parigi (1571-1635) in the years 1599-1600.

- Snell's Law and the Principle of Least Time

Euclid's Law of Reflection, formulated around 300 B.C., tells us what happens when a beam of light is reflected off a mirror. The Dutch physicist **Willebrord Snell** (1580 – 1626) was interested in the phenomenon of **refraction**, which is the change in direction that occurs when a beam of light crosses a boundary from one medium (say air) into another (say water). He observed that when the light beam enters the denser medium at some point  $C$  on the line, its velocity decreases and its path bends toward the normal to the line at  $C$ .

Snell's experiments led him to the discovery of the following relationship, known as **Snell's Law**:

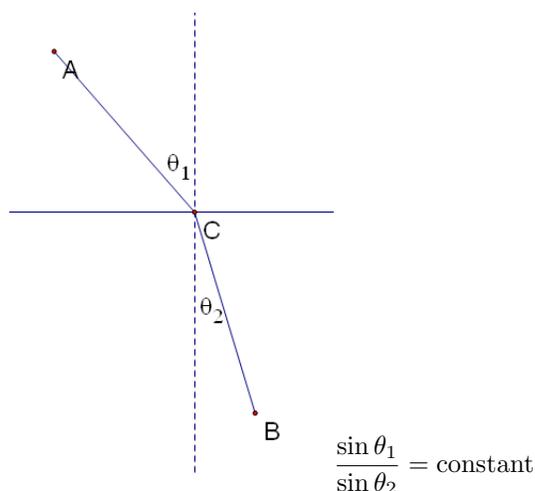


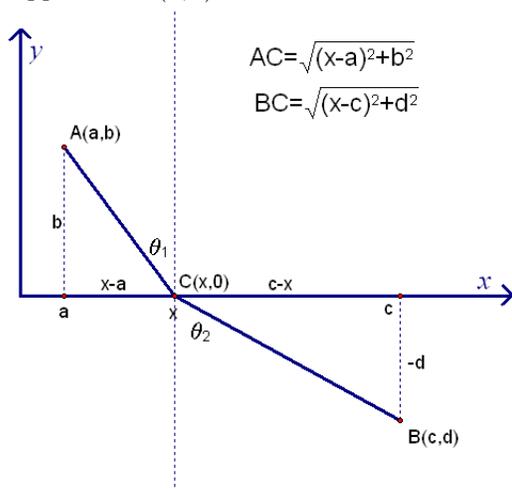
Figure 16: Snell's Law

Click [JSP here](#) to investigate Snell's Law. You can drag  $C$  to determine the approximate location of the point that minimizes the time of travel along the path  $A \rightarrow C \rightarrow B$ .

In 1637, **Fermat** became interested in finding a theoretical derivation of Snell's Law. Recall that Heron had shown that if a light beam traveled from  $A$  to  $B$  by first reflecting off a line  $\mathcal{L}$ , the path taken

was the one that would take the minimum time. Fermat reasoned that the same “least time” principle would govern that path from  $A$  to  $B$  across the boundary  $\mathcal{L}$ .

Fermat used the differential calculus (techniques which he himself developed by reasoning that the slope of a tangent line at a local maximum or minimum must be zero) to determine the quickest path. Suppose  $A = (a, b)$  lies above the  $x$  axis and  $B = (c, d)$  lies below the  $x$  axis, as in the picture below:



We wish to find the point  $C = (x, 0)$  on the  $x$  axis that minimizes the time of travel of a light beam along the path  $A \rightarrow C \rightarrow B$ , assuming its velocity above the  $x$  axis is  $v_1$ , and its velocity below the  $x$  axis is  $v_2$ . Since time =  $\frac{\text{distance}}{\text{velocity}}$ , the time of travel is

$$T(x) = \frac{\sqrt{(x-a)^2 + b^2}}{v_1} + \frac{\sqrt{(x-c)^2 + d^2}}{v_2}$$

If we take the derivative and set it equal to 0, we have

$$\frac{\frac{1}{2} [(x-a)^2 + b^2]^{-1/2} \cdot 2(x-a)}{v_1} + \frac{\frac{1}{2} [(x-c)^2 + d^2]^{-1/2} \cdot 2(x-c)}{v_2} = 0$$

which can be rewritten as

$$\frac{x-a}{v_1 \sqrt{(x-a)^2 + b^2}} + \frac{x-c}{v_2 \sqrt{(x-c)^2 + d^2}} = 0$$

But since  $\sin \theta_1 = \frac{x-a}{\sqrt{(x-a)^2 + b^2}}$  and  $\sin \theta_2 = \frac{c-x}{\sqrt{(x-c)^2 + d^2}}$ , the above equation can be rewritten as

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0$$

or equivalently,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

which is of course Snell's Law. But whereas Snell had conjectured his law based on observation, Fermat succeeded in giving a mathematical proof.

Fermat was led to investigate Snell's law after reading a paper by Descartes on the subject. Fermat thought Descartes' work did not make sense, and he could not understand how Descartes was able to arrive at the correct result using faulty and illogical arguments. Indeed, Fermat's criticisms could be interpreted as an accusation that Descartes, who made no mention of Snell in his work, had somehow gotten hold of Snell's experimental data, conjectured the correct law from the data, and then working backwards invented an argument that made little sense but led to the correct conclusion. Needless to say, Descartes did not take kindly to Fermat's accusations!

- L'Hôpital's Pulley Problem and the Principle of Least Potential Energy

The first calculus textbook was written by the [Marquis Guillaume-Francois-Antoine de L'Hôpital](#) (1661–1704). Best known for the famous rule that bears his name that is used to compute limits of indeterminate forms, L'Hôpital was very interested in mathematics and particularly in the new techniques of the calculus that had recently been developed by [Isaac Newton](#) (1643–1727) and [Gottfried Wilhelm Leibniz](#) (1646–1716). Consequently, he hired one of Leibniz's best students, [Johann Bernoulli](#) (1667–1748) to teach the calculus to him. L'Hôpital then took the notes he compiled from Bernoulli's lectures and organized them into a textbook called *Analyse des Infiniment Petits*. The book contained the original results of Newton, Leibniz, and Johann Bernoulli, as well as Johann's brother [Jakob Bernoulli](#) (1654–1705). L'Hôpital has been justly criticized for giving little or no credit to these mathematicians, presenting the mathematics as if it were his own. Indeed, many have suggested that L'Hôpital's Rule should really be called Bernoulli's Rule, since it was taught to him by Johann! Johann Bernoulli was forced to keep silent during L'Hôpital's lifetime since L'Hôpital had essentially paid him for the right to use Bernoulli's results in any way he wanted. After L'Hôpital's death, however, Bernoulli began to take credit for practically the entire book.

Being a textbook, *Analyse des Infiniment Petits* contains a number of interesting problems that are used to demonstrate the techniques of the new calculus. One of these problems is the famous pulley problem, illustrated below.

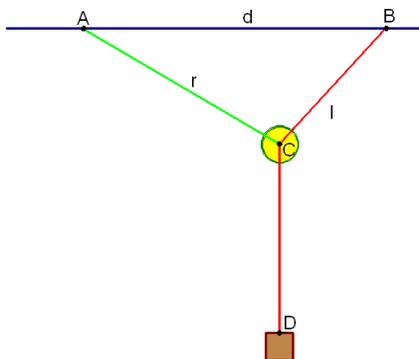


Figure 17: L'Hôpital's Pulley Problem

A (weightless) cable of length  $r$  is attached to the ceiling at point  $A$ . The other end of the cable is attached to a pulley (point  $C$ ). The end of another cable of length  $l$  is attached to the ceiling at point  $B$ , where  $|AB| = d$  ( $d > r$ ). At the other end of the second cable (point  $D$ ) is attached a weigh. This second cable passes over the pulley (so  $|BC| + |CD| = l$ ), and the whole system is allowed to adjust itself to reach a state of stationary equilibrium. The question is: How far below the ceiling is the final position of the weight? The solution relies on the **principle of minimum potential energy**. This means that the system will be stable when its potential energy is the least, i.e. when the weight hangs as far below the ceiling as possible. Click [JSP here](#) to explore L'Hôpital's Pulley Problem and identify the approximate equilibrium position for the example shown.

- Regiomontanus' Hanging Picture Problem

[Johann Müller](#) (1436–1476), commonly known today as Regiomontanus, was perhaps the most influential mathematician of the fifteenth century. Although his contributions were mainly in the area of trigonometry, we are interested here in the following problem that he posed in a letter he wrote in 1471: “At what point on the ground does a perpendicularly suspended rod appear largest?” In other words, which point  $P$  on line  $\mathcal{L}$  in the picture below maximizes the angle  $APB$ ?

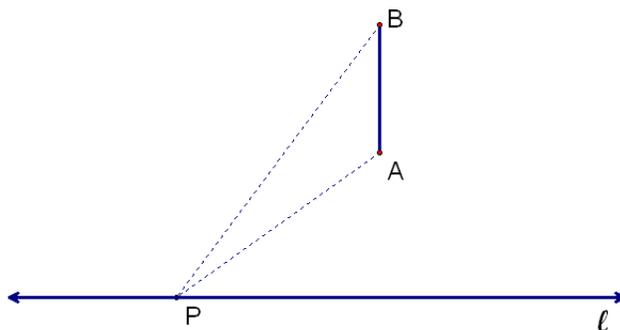


Figure 18: Regiomontanus' problem: Which point  $P$  on  $\mathcal{L}$  maximizes  $\angle APB$ ?

Click [JSP here](#) to explore Regiomontanus' Hanging Picture Problem and identify the point  $P$  that maximizes  $\angle APB$  for the example shown. This problem is the first optimization problem that appears in the history of mathematics since the days of antiquity. It was probably inspired by questions of perspective that Renaissance artists of that time period were grappling with. It is often stated in modern calculus textbooks as follows: "A painting is hung flat against a wall. How far from the wall should one stand to maximize the viewing angle subtended at his eye by the painting?"

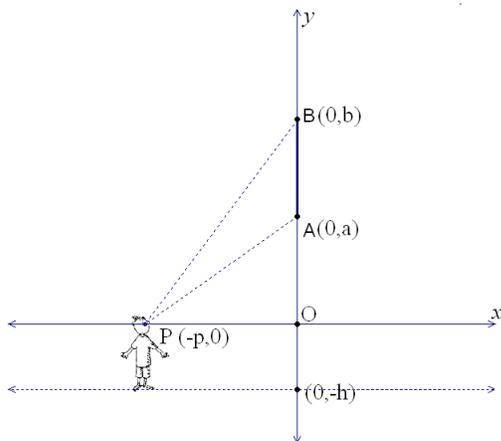


Figure 19: Modern equivalent of Regiomontanus' problem: How far from a wall should observer stand to maximize his viewing angle of a picture on the wall?

In the above picture, the line  $\mathcal{L}$  is drawn at the height of the viewer's eye. We can impose a coordinate system on the above picture, with  $\mathcal{L}$  as the  $x$  axis, and  $A$  and  $B$  on the  $y$  axis, say  $A = (0, a)$  and  $B = (0, b)$ . Then the height of the painting is  $b - a$ , and the bottom edge of the painting is  $a + h$  units above the floor, where  $h$  is the distance above the floor of the viewer's eye.

The original solution to this problem, like Heron's solution to the shortest distance problem, is geometric and does not use calculus. Consider a circle that goes through points  $A$  and  $B$  and is tangent to the line  $\mathcal{L}$ . Let  $P$  be the point of tangency. Then  $P$  is the point on  $\mathcal{L}$  that maximizes angle  $APB$ .

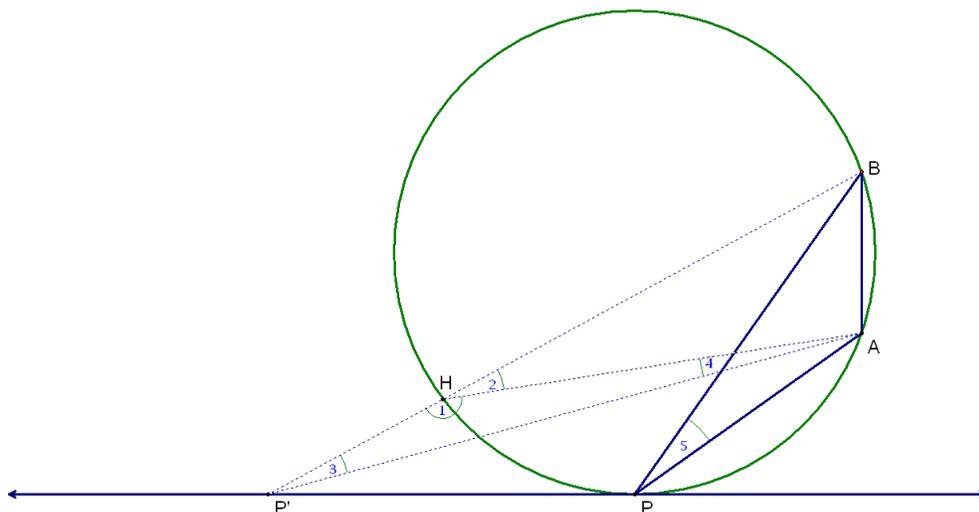


Figure 20: Geometric solution to Regiomontanus' Problem:  $P$  should be such that line  $\mathcal{L}$  is tangent to the circle through  $P$ ,  $A$ , and  $B$

To see this, let  $P'$  be another point on  $\mathcal{L}$ . We claim that  $\angle 5 > \angle 3$ . Let  $H$  be the point of intersection of the circle and  $BP'$ . Then  $\angle 2 = \angle 5$  since both angles subtend the same chord. Also,  $\angle 1 + \angle 2 = \angle 1 + \angle 3 + \angle 4$ , since both equal 180 degrees. Subtracting  $\angle 1$  from both sides gives  $\angle 2 = \angle 3 + \angle 4$ . Hence

$$\angle 5 = \angle 2 = \angle 3 + \angle 4 > \angle 3$$

as was to be shown.

- Galileo and the Brachistochrone Problem

The last optimization problem that we discuss here is one of the most famous problems in the history of mathematics and was posed by the Swiss mathematician John Bernoulli in 1696 as a challenge “to the most acute mathematicians of the entire world”. The problem can be stated as follows: Given two points on a plane at different heights, what is the shape of the wire down which a bead will slide (without friction) under the influence of gravity so as to pass from the upper point to the lower point in the shortest amount of time?”

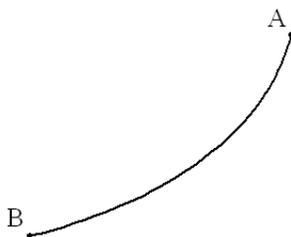
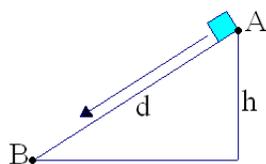


Figure 21: Brachistochrone Problem: Which path from  $A$  to  $B$  is traversed in the shortest time?

This is the brachistochrone (“shortest time”) problem. This was a different kind of optimization problem, since instead of asking for the value of a variable, among all possible values, that will maximize or minimize something, it asks for the optimal function (or curve), among all possible curves. The differential calculus does not provide all the tools necessary to solve this problem. So for the moment,

we would like to discuss Galileo's work relevant to this problem, which occurred in 1638, well before the brachistochrone problem had been explicitly stated.

Galileo studied motion under gravity, showing that a body falling in space traverses a distance proportional to the square of the time of descent. Using this law, he was able to compute the time of descent of an object falling along an inclined plane from point  $A$  to point  $B$ , assuming no friction.

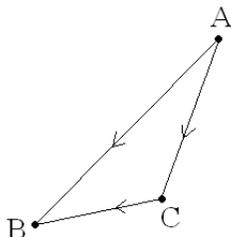


$$\text{Time to travel from } A \text{ to } B \text{ is } \sqrt{\frac{2}{g}} \frac{d}{\sqrt{h}} \quad (1)$$

Figure 22: Galileo's formula for time of descent along inclined plane

In equation (1),  $d$  is the distance between points  $A$  and  $B$ ,  $h$  is the vertical distance between  $A$  and  $B$ , and  $g$  is the acceleration due to gravity, approximately  $980 \text{ cm/sec}^2$ .

Clearly the shortest path from  $A$  to  $B$  is a straight line, but is this path the one that will take the shortest time? For example, is there some point  $C$  such that if the body were to fall following a straight line from  $A$  to  $C$ , and then a straight line from  $C$  to  $B$ , it would do so in less time than if it traversed a straight-line path from  $A$  to  $B$ ?



Galileo appeared to believe that the answer to the brachistochrone problem was a circle. Click [JSP here](#) to explore this problem and compare the times of descent along different polygonal paths. Was Galileo correct?

- [Module 3 Exercises](#)

- [Geometer's Sketchpad file needed for Module 3](#)



## References

- [1 ] Paul J. Nahin, *When Least is Best: How Mathematicians Discovered Many Clever Ways to Make Things as Small (or as Large) as Possible*, Princeton University Press, 2004.
- [2 ] V.M. Tikhomirov, *Stories about maxima and minima*; translated from the Russian by Abe Shenitzer, American Mathematical Society/Mathematical Association of America, 1990.

## Module 1: Ellipse Drawers

### Exercises

#### EXERCISE 1.

The ancient Greeks defined an ellipse as the set of points  $P$  such that the sum of the distances from  $P$  to the two fixed points  $F$  and  $F'$  (called the foci) is always constant. Probably one of the oldest ways of drawing an ellipse is the string construction shown below, in which two ends of a string are attached to two fixed pins  $F$  and  $F'$ , and then a pen holds the string taut as it rotates around the pins. The first person known to have written about the string construction of the ellipse was [Anthemius of Tralles](#) in the fifth century, although it must certainly have been used earlier.

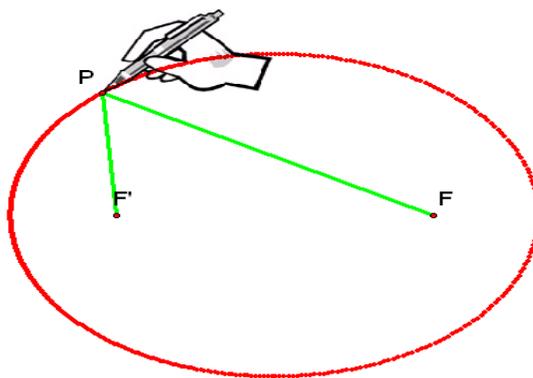


Figure 23: String Construction of Ellipse

- (a) Assume a coordinate system with  $F$  and  $F'$  on the  $x$  axis and the origin half way between  $F$  and  $F'$ , as shown below:

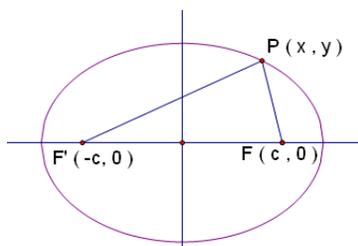


Figure 24: Obtaining an Algebraic Equation of the Ellipse

Then the coordinates of  $F$  and  $F'$  are  $(\pm c, 0)$  for some  $c$ . If  $P = (x, y)$  is a point on the ellipse, then we must have  $|PF| + |PF'| = d$  where  $d$  is a constant (the length of the string in the string construction). Use the distance formula to translate  $|PF| + |PF'| = d$  into an algebraic equation involving  $x$ ,  $y$ ,  $c$ , and  $d$ .

*Solution:*  $\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = d$  □

- (b) Use a CAS (i.e. a Computer Algebra System such as Derive, Maple, or Mathematica, a TI Voyage 200 calculator, or a TI 89 graphing calculator) to solve the equation for  $y$ . You should get two equations, one for the top half and one for the bottom half. Note: If your CAS gives you a formula that involves the product of two square roots, rewrite it so that there's only one square root.

*Solution:*  $y = \frac{\sqrt{(4c^2 - d^2) \cdot (4x^2 - d^2)}}{2 \cdot |d|}$  or  $y = -\frac{\sqrt{(4c^2 - d^2) \cdot (4x^2 - d^2)}}{2 \cdot |d|}$  □

- (c) Use the equation to express the  $x$  intercepts of the ellipse in terms of  $c$  and/or  $d$ .

*Solution:*  $x$  intercepts occur when  $y = 0 \Rightarrow 4x^2 - d^2 = 0 \Rightarrow x = \pm \frac{d}{2}$  □

- (d) Use the equation to express the  $y$  intercepts of the ellipse in terms of  $c$  and/or  $d$ .

*Solution:*  $y$  intercepts occur when  $x = 0 \Rightarrow y = \pm \frac{\sqrt{(4c^2 - d^2) \cdot (0 - d^2)}}{2 \cdot |d|} \Rightarrow \boxed{y = \pm \frac{\sqrt{d^2 - 4c^2}}{2}}$  □

- (e) What values of  $c$  and  $d$  will generate an ellipse with  $x$  intercepts at  $(\pm 5, 0)$  and  $y$  intercepts at  $(0, \pm 3)$ ? Write the equations of the ellipse.

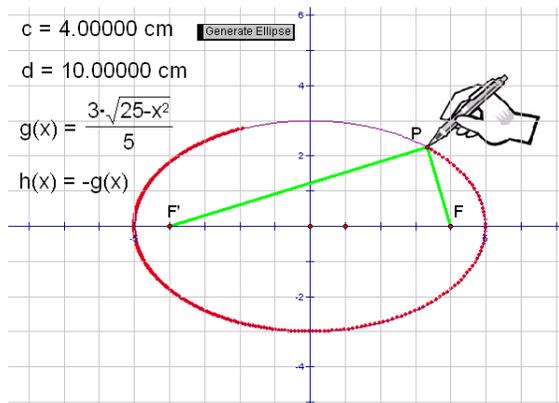
*Solution:*  $\frac{d}{2} = 3 \Rightarrow \boxed{d = 6}$ ;

$$\frac{\sqrt{d^2 - 4c^2}}{2} = 5 \Rightarrow d^2 - 4c^2 = 36 \Rightarrow 4c^2 = d^2 - 36 = 36 - 36 = 0 \Rightarrow c^2 = 0 \Rightarrow \boxed{c = 0}$$

The equations are  $\boxed{y = \pm \frac{3\sqrt{25 - x^2}}{5}}$  □

- (f) The string ellipse construction is illustrated on Page 1 of the Geometer's Sketchpad file *EllipseDrawers.gsp*.

- To introduce a coordinate system, you first need to construct the origin, which should be half way between  $F$  and  $F'$ . To do this, select the points  $F$  and  $F'$  then choose **Construct**  $\gg$  **Segment**. Then select this segment and choose **Construct**  $\gg$  **Midpoint**. Now select this point and choose **Graph**  $\gg$  **Define Origin**.
- Double-click on the parameter  $c$  to change its value to the one you found in part (e). Also change the value of  $d$ . Then click on the "Generate Ellipse" button to generate an ellipse. If you computed  $c$  and  $d$  correctly, your ellipse should have  $x$  intercepts  $(\pm 5, 0)$  and  $y$  intercepts  $(\pm 3, 0)$ .
- Choose **Graph**  $\gg$  **Plot New Function** and enter one of the equations you found in part (b). Repeat to enter the other equation. The resulting graphs should match part of the curve generated by the ellipse drawer. You can erase traces (ctrl-B) and click on the "Generate Ellipse" button again to redraw the ellipse and verify that it exactly matches the graph of your function.



*Solution:* □

**EXERCISE 2.** The Dutch mathematician Franz Van Schooten (1615–1660) came up with the following simple device for drawing ellipses. As the point  $E$  is dragged along  $AB$ , the point  $P$  traces out the curve.

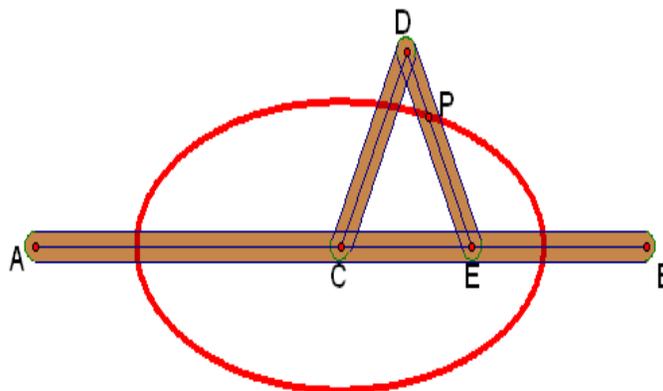
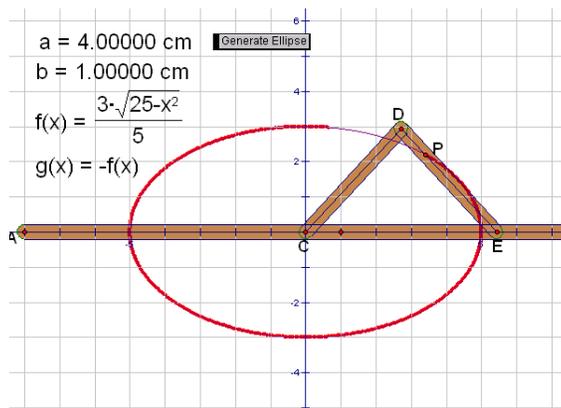


Figure 25: Franz Van Schooten's Ellipse Drawer

- (a) Assume a coordinate system with origin at  $C$  and  $x$  axis through  $A$  and  $B$ . What should the lengths of  $a = CD$  and  $b = DP$  be so that the ellipse generated will have  $x$  intercepts at  $(\pm 5, 0)$  and  $y$  intercepts at  $(0, \pm 3)$ ?

*Solution:* This device will generate an ellipse with  $x$  intercept  $CD + DP$  and  $y$  intercept  $PE$ . Hence we need to solve  $a + b = 5$  and  $a - b = 3$  for  $a$  and  $b$ . Adding the two equations we obtain  $2a = 8$  or  $\boxed{a = 4}$ . Then  $\boxed{b = 1}$  □

- (b) Van Schooten's ellipse drawer is illustrated on Page 2 of the Geometer's Sketchpad file EllipseDrawers.gsp.
- Introduce a coordinate system with origin at  $C$ . Change the values of the parameters  $a$  and  $b$  to the ones you found above, then click on the "Generate Ellipse" button to generate an ellipse. If you computed  $a$  and  $b$  correctly, your ellipse should have  $x$  intercepts  $(\pm 5, 0)$  and  $y$  intercepts  $(\pm 3, 0)$ .
  - Go to Page 1 (the string construction), select the parameters  $c$  and  $d$  and the two functions, then choose **Edit**  $\gg$  **Copy**. Then go to Page 2 (Van Schooten's ellipse drawer) and choose **Edit**  $\gg$  **Paste**. Finally select the two functions and choose **Graph**  $\gg$  **Plot Functions**. The graphs should again match the curve generated by the ellipse drawer (you may need to redraw it by clicking the "Generate Ellipse" button again).



*Solution:* □

**EXERCISE 3.** Proclus' (418–485) ellipse drawer generated an ellipse by tracing a point inside a circle as it rolled without slipping inside and tangent to another circle whose radius was twice as long.

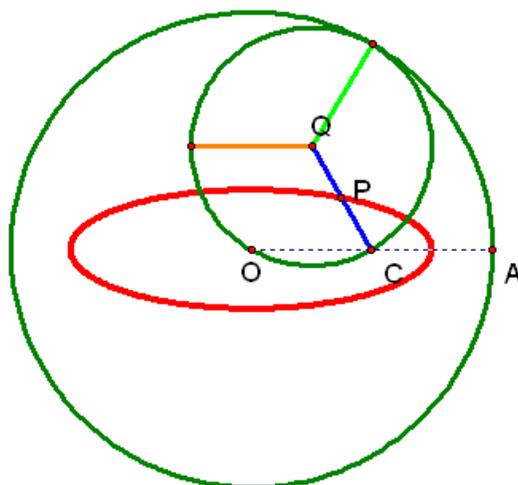
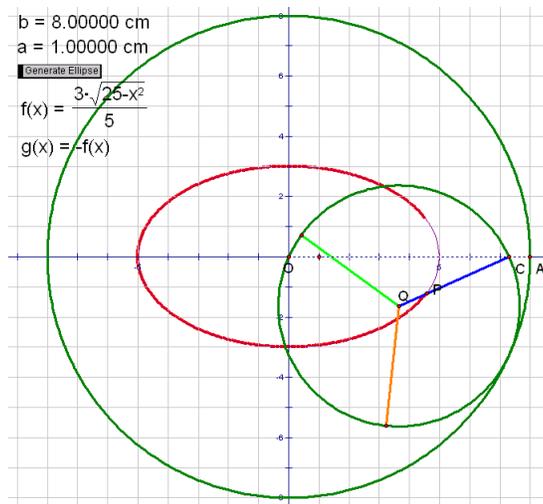


Figure 26: Proclus' Ellipse Drawer

- (a) Assume a coordinate system with center at  $O$  and  $x$ -axis through  $O$  and  $A$ . If  $OA = b$ ,  $QC = \frac{b}{2}$ , and  $QP = a$ , find the values of  $a$  and  $b$  that will generate an ellipse with  $x$  intercepts at  $(\pm 5, 0)$  and  $y$  intercepts at  $(0, \pm 3)$ .

*Solution:* From the construction we must have  $\frac{b}{2} + a = 5$  and  $\frac{b}{2} - a = 3$ , so  $b = 8$  and  $a = 1$   $\square$

- (b) Proclus' ellipse drawer is illustrated on Page 3 of the Geometer's Sketchpad file EllipseDrawers.gsp.
- Introduce a coordinate system with origin at  $O$ . Change the values of the parameters  $a$  and  $b$  to the ones you found above, then click on the "Generate Ellipse" button to generate an ellipse. If you computed  $a$  and  $b$  correctly, your ellipse should have  $x$  intercepts  $(\pm 5, 0)$  and  $y$  intercepts  $(\pm 3, 0)$ .
  - As before, copy/paste the functions from Page 1 to Page 3 and graph them. The graphs should again match the curve generated by the ellipse drawer.



*Solution:*  $\square$

**EXERCISE 4.** Finally, an ellipse drawer attributed to Archimedes is the trammel construction shown below.

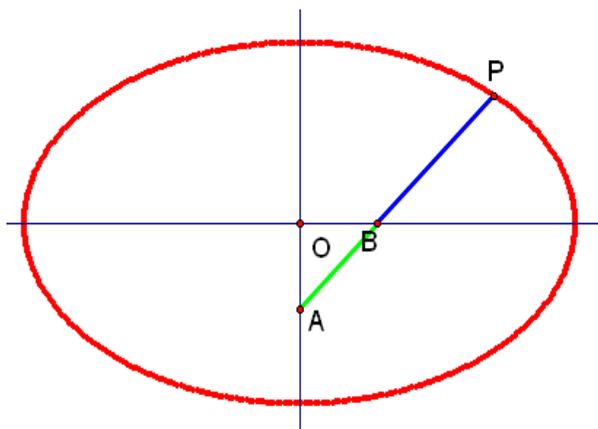


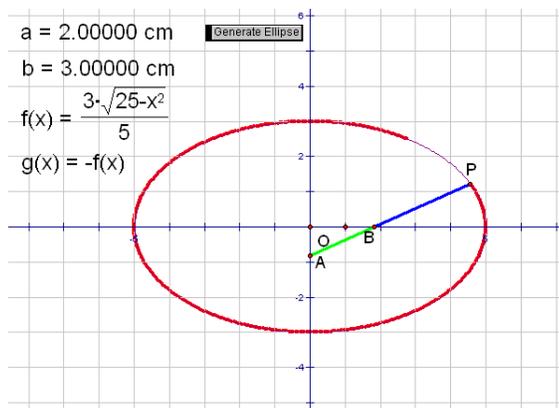
Figure 27: Archimedes' Trammel

The point  $B$  on segment  $AP$  is attached to the horizontal axis in such a way that it can slide along the axis. Similarly, the point  $A$  is attached to the vertical axis and allowed to slide along it. The point  $P$  then traces out the curve.

- (a) Assuming a coordinate system with origin at  $O$ , determine the values of  $a = AB$  and  $b = BP$  such that the ellipse generated by Archimedes' trammel will have  $x$  intercepts at  $(\pm 5, 0)$  and  $y$  intercepts at  $(0, \pm 3)$ .

*Solution:* We must have  $AP = 5$  and  $BP = 3$ . Hence  $b = 3$  and  $a = 2$  □

- (b) Archimedes' ellipse drawer is illustrated on Page 4 of the Geometer's Sketchpad file `EllipseDrawers.gsp`.
- Introduce a coordinate system with origin at  $O$ . Change the values of the parameters  $a$  and  $b$  to the ones you found above, then click on the "Generate Ellipse" button to generate an ellipse. If you computed  $a$  and  $b$  correctly, your ellipse should have  $x$  intercepts  $(\pm 5, 0)$  and  $y$  intercepts  $(\pm 3, 0)$ .
  - As before, copy/paste the functions from Page 1 to Page 3 and graph them. The graphs should again match the curve generated by the ellipse drawer.



*Solution:*

□

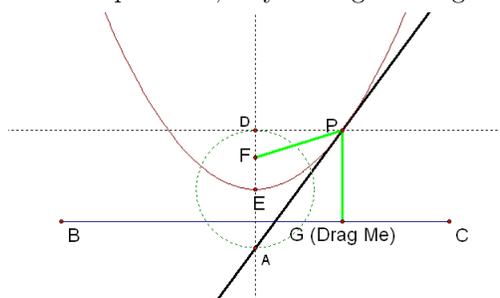
## Module 2: Drawing Tangent Lines from Apollonius to Fermat

### Exercises

#### EXERCISE 5.

In this exercise, you will use Geometer's Sketchpad to follow Apollonius' recipes for construction of tangents to conics. Open the Geometer's Sketchpad file Tangents.gsp.

- (a) Page 1 of the file Tangents.gsp contains a construction of a parabola, the set of points equidistant from the point  $F$  and the line  $BC$ . Use the recipe provided by Apollonius in Proposition I-33 to construct a tangent line at  $P$ . To do this, you will need to construct the axis of symmetry, the point  $D$ , the point  $A$  so that  $AE = ED$ , and finally the tangent line  $AP$ , as in Figure 4. Your construction should work at every point on the parabola, as you drag  $G$  along  $AB$ .



*Solution:*

□

- (b) Page 2 of the file Tangents.gsp contains a construction of an ellipse, using Van Schooten's ellipse drawer. Use the recipe provided by Apollonius in Proposition I-34 to construct a tangent line at  $P$ . [Hint: Note that, for example when  $P$  is on the left half of the ellipse as in Figure 5(a),  $|AH| = |AG| + |GH|$ ; make this substitution in the proportion given in Proposition I-34 and solve for  $AG$ . Use the "Measure Distance" and "Measure Calculate" menus to compute  $AG$ . Now construct  $A$  using the formula you obtained as follows:

1. Select the calculated measurement  $AG$  and choose **Transform**  $\gg$  **Mark Distance**.
2. Select the point  $G$  and choose **Transform**  $\gg$  **Translate**. Select a polar translation vector with Marked Distance and Fixed Angle  $\pi$ . Click Translate. The resulting point is  $A$ . Measure  $AG$  to verify that the value agrees with your calculation.

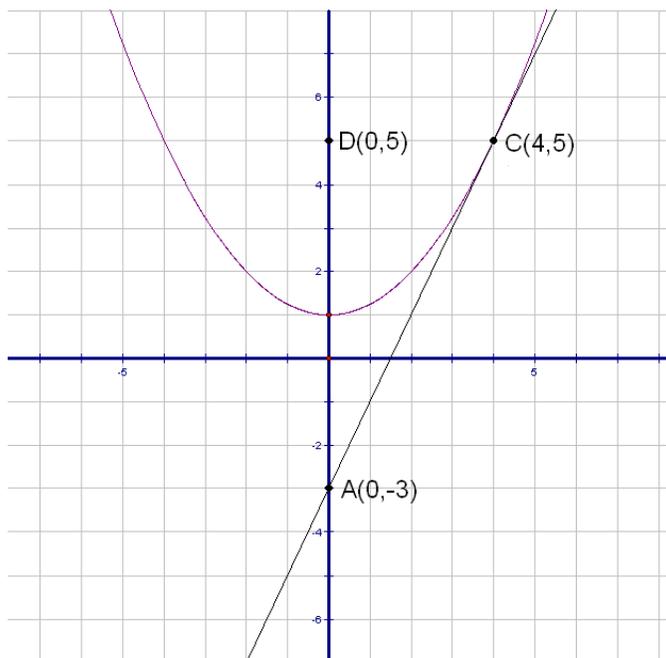
Finally construct the tangent line  $PA$ .]

*Solution:* We are given  $\frac{AH}{AG} = \frac{BH}{BG}$ . Substituting we have

$$\begin{aligned} \frac{BH}{BG} &= \frac{AG + GH}{AG} = 1 + \frac{GH}{AG} \\ \Rightarrow \frac{GH}{AG} &= \frac{BH}{BG} - 1 = \frac{BH - BG}{BG} \\ \Rightarrow \frac{AG}{GH} &= \frac{BG}{BH - BG} \\ \Rightarrow AG &= \frac{BG \cdot GH}{BH - BG} \end{aligned}$$

We use the above formula to compute  $AG$ , then construct  $A$  on the horizontal axis  $AG$  units from  $G$ .



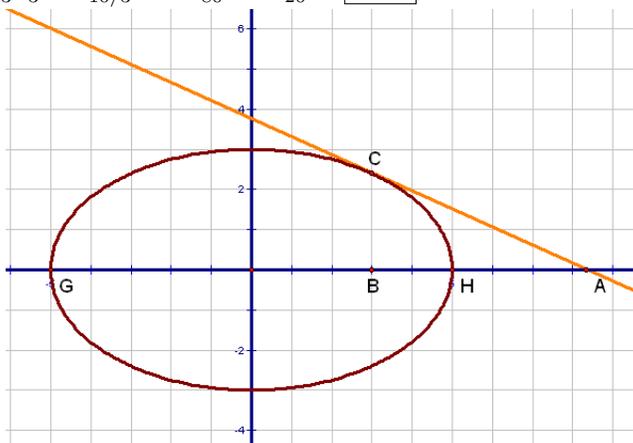


□

- (b) Use Apollonius' method to find the  $x$  intercept of the tangent line to the ellipse  $\frac{x^2}{25} + \frac{y^2}{9} = 1$  at the point  $C(3, \frac{12}{5})$ . The ellipse is shown in Figure 28(b). Carefully draw the tangent line and determine its slope.

*Solution:*  $G = (-5, 0)$ ,  $H = (5, 0)$ ,  $B = (3, 0)$ , and  $\frac{AH}{AG} = \frac{BH}{BG} = \frac{2}{8} = \frac{1}{4}$ . Since  $AH + GH = AG$ , we have  $\frac{AG - GH}{AG} = \frac{1}{4} \Rightarrow 1 - \frac{GH}{AG} = \frac{1}{4} \Rightarrow \frac{GH}{AG} = \frac{3}{4}$ , and so  $3AG = 4GH = 40$ . Hence  $AG = \frac{40}{3}$ . The  $x$  coordinate of  $A$  then is  $\frac{40}{3} - 5 = \frac{25}{3} = \boxed{8\frac{1}{3}}$ .

The slope of the tangent line is the slope of the line through  $A(25/3, 0)$  and  $C(3, 12/5)$ , which is  $\frac{0 - 12/5}{25/3 - 3} = \frac{-12/5}{16/3} = -\frac{36}{80} = -\frac{9}{20} = \boxed{-.45}$ .

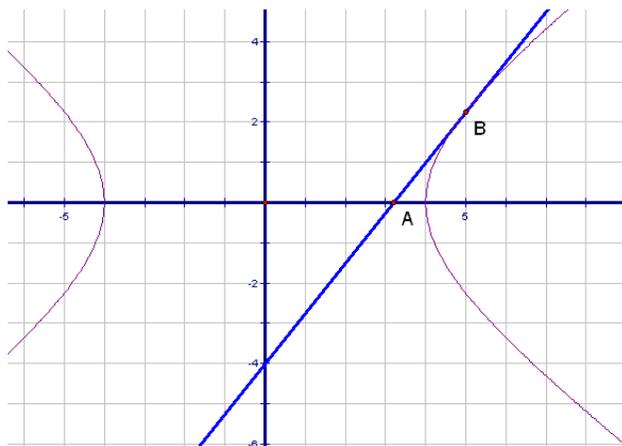


□

- (c) Use Apollonius' method to find the  $x$  intercept of the tangent line to the hyperbola  $\frac{x^2}{16} - \frac{y^2}{9} = 1$  at the point  $(5, \frac{9}{4})$ . The hyperbola is shown in Figure 28(c). Carefully draw the tangent line, then use your sketch to determine the equation of the tangent line.

*Solution:*  $G = (-4, 0)$ ,  $H = (4, 0)$ ,  $B = (5, 0)$ , and  $\frac{AH}{AG} = \frac{BH}{BG} = \frac{1}{9}$ . Since  $AH + AG = 8$ , we have  $\frac{AH}{8 - AH} = \frac{1}{9} \Rightarrow 9AH = 8 - AH \Rightarrow 10AH = 8 \Rightarrow AH = .8$ , and so the  $x$  coordinate of  $A$  then is  $4 - .8 = 3.2$  so  $A = (3.2, 0)$ .

The slope of the tangent line is the slope of the line through  $A(3.2, 0)$  and  $C(5, 9/4)$ , which is  $\frac{0 - 2.25}{3.2 - 5} = \frac{-2.25}{-1.8} = \boxed{1.25}$ .

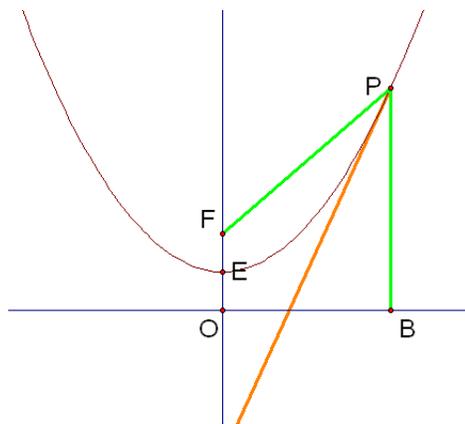


□

## EXERCISE 7.

- (a) Open page 3 (Parabola) of the file Tangents.gsp, which contains the construction of the parabola as the locus of points  $P$  that are equidistant from a fixed point  $F$  (the focus) and a fixed line  $OB$  (the directrix). You can see that as you drag the point labeled “Drag Me”, the distances  $BP$  and  $PF$  are always equal.

There are two forces acting on  $P$ : one toward  $F$ , and one toward the horizontal line  $OB$ . Since  $|FP| = |BP|$ , the two forces are equal (if  $P$  moves a little toward  $F$ , it must move by the same amount toward the line). Hence the direction of motion should be the angle bisector of  $\angle FPB$ . Construct the angle bisector to verify that this does indeed appear to be tangent to the curve at every point of the parabola.

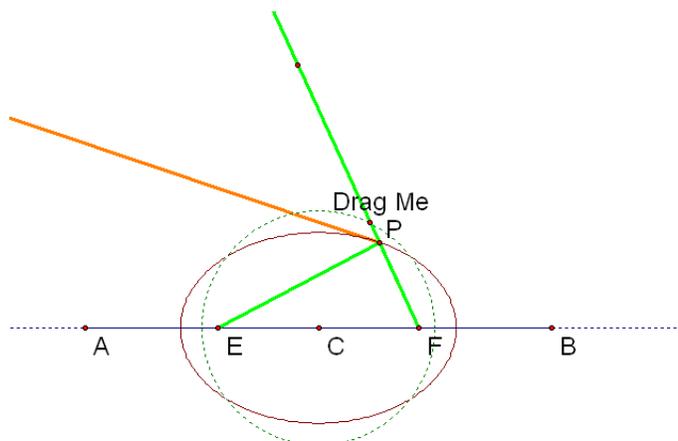


*Solution:*

□

- (b) On page 4 (Ellipse) of the Geometer’s Sketchpad file Tangents.gsp is the construction of the ellipse as the locus of points  $P$  such that the sum of  $P$ ’s distances from two fixed points  $F$  and  $F'$  is constant. Note that as you drag the point labeled “Drag Me”, the sum  $|PF| + |PF'|$  remains constant.

Construct the tangent to the ellipse at any point  $P$ . [Hint: Since  $|PF| + |PF'|$  is constant, each time  $P$  moves toward  $F$  it must move the same distance *away from*  $F'$ . So there are two equal forces acting on  $P$ , one toward one of the foci, the second away from the other focus.]

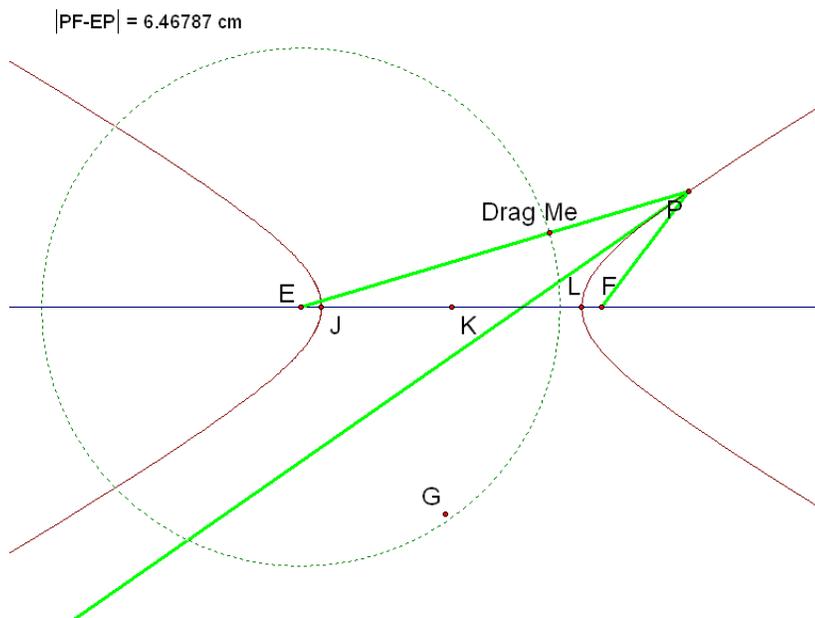


*Solution:*

□

- (c) On page 5 (Hyperbola) of the Geometer's Sketchpad file Tangents.gsp is the construction of the hyperbola as the locus of points  $P$  such that the difference of  $P$ 's distances from two fixed points  $F$  and  $F'$  is constant. Note that as you drag the point labeled "Drag Me", the difference  $|PF - PF'|$  remains constant.

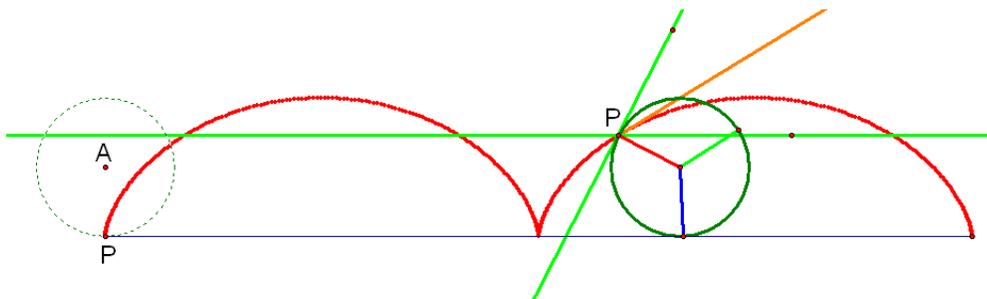
Construct the tangent to the hyperbola at any point  $P$ . Make sure your construction(s) work at every point of the hyperbola.



*Solution:*

□

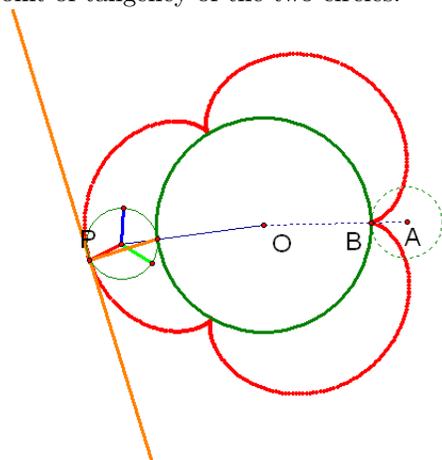
- (d) The cycloid, illustrated on page 6 of the Geometer's Sketchpad file Tangents.gsp, played a very important part in the history of the calculus, and its properties were studied extensively during the sixteenth and seventeenth centuries. It is the path traced out by a point on the circumference of a circle as the circle rolls on a horizontal line. Use Roberval's method to construct the tangent line at any point  $P$  of the cycloid. [Hint: At the instant when the rolling circle touches the horizontal line at a point  $A$ , the point  $P$  is rotating about  $A$ , so  $P$  is moving in a direction perpendicular to  $AP$ .]



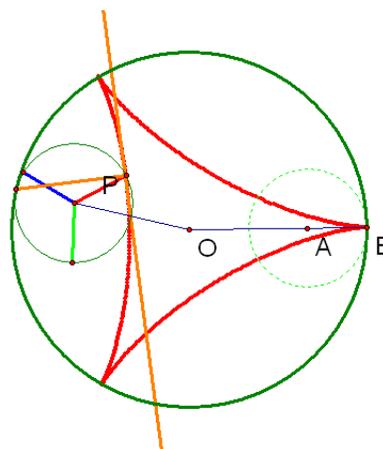
*Solution:*

- (e) (Extra Credit) An epicycloid is the path traced out by a point on the circumference of a circle as the circle rolls outside and along the circumference of another circle. A hypocycloid is the path traced out by a point on the circumference of a circle as the circle rolls inside and along the circumference of another (larger) circle. These curves are illustrated on pages 7 and 8 of Tangents.gsp. You can change the relative size of the circles by changing the value of  $m$ . Use Roberval's method to construct the tangent at any point to any epicycloid and to any hypocycloid.

*Solution:* The tangent line in both cases is constructed as a line perpendicular to  $AP$  at  $P$  where  $A$  is the point of tangency of the two circles. □



Epicycloid with  $m = 3$



Hypocycloid with  $m = 3$  □

**EXERCISE 8.** In this exercise you will use Descartes' method to find tangent lines to the curve  $y = \sqrt{4x}$ . The method consists of first finding a circle tangent to the curve. Then the tangent line to the curve will be the same as the tangent line to the circle, which is easily constructed since it is perpendicular to the radius.

- (a) Open page 9 of the file Tangents.gsp. This page contains a graph of  $y = \sqrt{4x}$ , the point  $A = (1, 2)$  which is on this curve, a point  $O$  on the  $x$  axis, and a circle with center  $O$  and radius  $OA$ . In general, depending on where point  $O$  is, this circle will cross the curve at two points  $A$  and  $B$ . Now drag  $O$  along the  $x$  axis until the two points of intersection  $A$  and  $B$  merge into one. The resulting circle touches the curve at a single point and so is tangent to the curve.

1. The  $x$  coordinate of  $O$  is given by the measurement  $x_O$ . What is the  $x$  coordinate of the center of the circle of tangency at the point  $A(1, 2)$ ? [Hint: It should be an integer.] □

*Solution:*  $x = 3$

2. Compute the slope of the radius  $OA$ . □

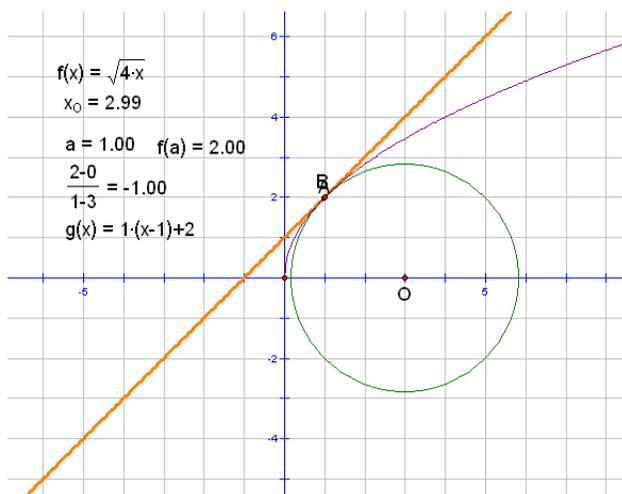
*Solution:*  $\frac{2-0}{1-3} = -1$

3. We know that the tangent line to the circle at  $A$  is perpendicular to the radius  $OA$ . Compute the slope of the tangent line. □

*Solution:*  $-\frac{1}{-1} = 1$

4. Compute the equation of the tangent line, then graph it. □

*Solution:*  $y = 1 \cdot (x - 1) + 2$  or  $y = x + 1$ .



□

(b) The  $x$  coordinate of the point  $A$  on page 9 of the file Tangents.gsp can be changed by double-clicking on the parameter  $A$ . Let  $A = (9, 6)$ . Again, drag  $O$  until you obtain a circle tangent to the curve at  $A$ .

1. What is the  $x$  coordinate of the center of the circle of tangency? [Hint: Again, the answer is an integer.]

*Solution:*  $x = 11$

□

2. Compute the slope of the radius  $OA$ .

*Solution:*  $\frac{6-0}{9-11} = -3$

□

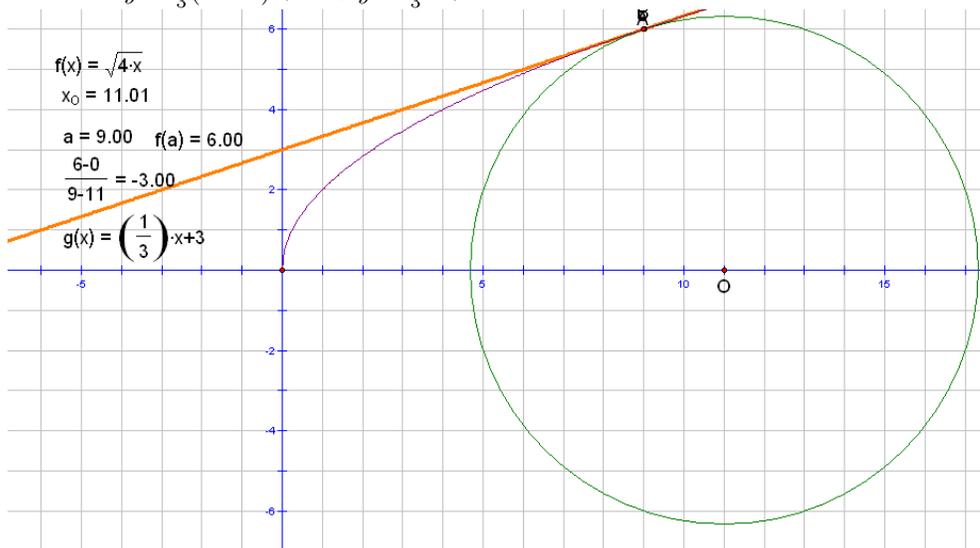
3. Compute the slope of the tangent line.

*Solution:*  $-\frac{1}{-3} = \frac{1}{3}$

□

4. Compute the equation of the tangent line, then graph it.

*Solution:*  $y = \frac{1}{3}(x - 9) + 6$  or  $y = \frac{1}{3}x + 3$



□

### Module 3: Optimization Problems Then and Now

#### Exercises

##### EXERCISE 9.

- (a) Use calculus and a CAS to find the point  $C$  on the  $x$  axis for which  $|AC| + |CB|$  is the shortest, where  $A = (0, 4)$  and  $B = (10, 12)$ .

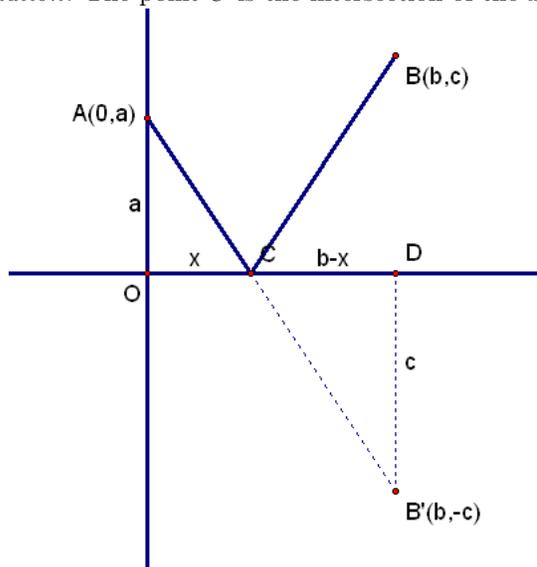
*Solution:* If  $C = (x, 0)$ , then we want to minimize  $f(x) = |AC| + |CB| = \sqrt{(0-x)^2 + (4-0)^2} + \sqrt{(10-x)^2 + (12-0)^2} = \sqrt{x^2 - 20x + 244} + \sqrt{x^2 + 16}$ . We use a CAS to compute the derivative:

$$f'(x) = \frac{x\sqrt{x^2 - 20x + 244} + (x - 10)\sqrt{x^2 + 16}}{\sqrt{x^2 + 16}\sqrt{x^2 - 20x + 244}}$$

Then  $f'(x) = 0 \Rightarrow x = \frac{5}{2}$ . So the point  $C$  that minimizes  $|AC| + |CB|$  is  $C = (\frac{5}{2}, 0)$ .  $\square$

- (b) Use Heron's method to find a formula for the  $x$  coordinate of the point  $C$  on the  $x$  axis that minimizes  $|AC| + |CB|$  if  $A = (0, a)$  and  $B = (b, c)$ . Use your formula to verify your answer to part (a).

*Solution:* The point  $C$  is the intersection of the  $x$  axis and segment  $AB'$  where  $B'$  is the point  $(b, -c)$ .



In the above picture, the triangles  $AOC$  and  $B'DC$  are similar, so we can solve the proportion  $\frac{a}{x} = \frac{c}{b-x}$  for  $x$  to obtain  $x = \frac{ab}{a+c}$ .

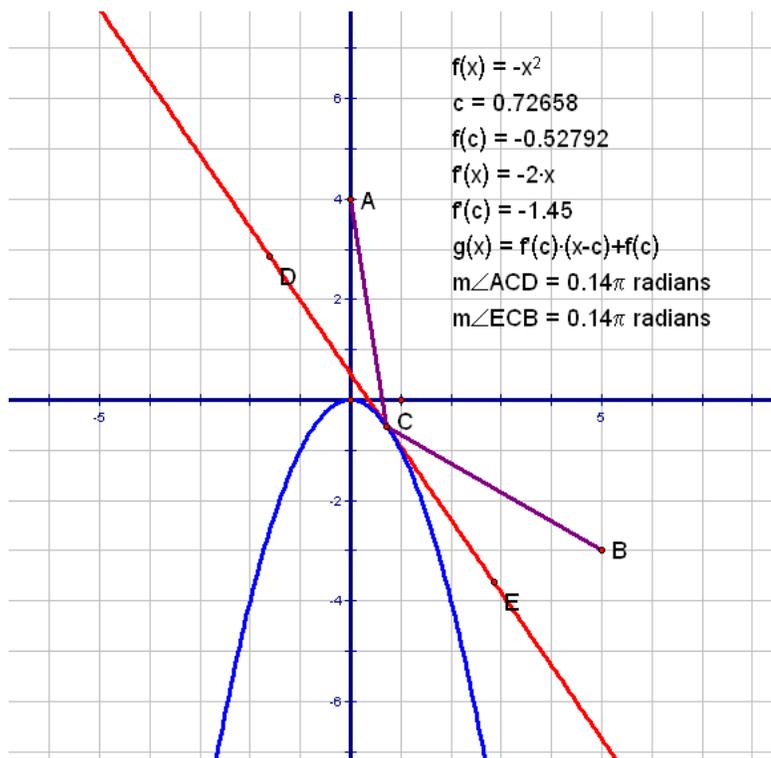
In part (a), we were given  $a = 4$ ,  $b = 10$  and  $c = 12$ . So  $x = \frac{40}{16} = \frac{5}{2}$  which is the answer we obtained using calculus.  $\square$

- (c) Use calculus and a CAS to find the coordinates of the point  $C$  on the curve  $y = -x^2$ , so that  $|AC| + |BC|$  is a minimum, where  $A = (0, 4)$ , and  $B = (5, -3)$ . Illustrate your solution on Geometer's Sketchpad. Your sketch should include the following:

1. a plot of the curve  $f(x) = -x^2$
2. a plot of the two points  $A$  and  $B$
3. a plot of the point  $C$  that solves the problem
4. a graph of the tangent line to the curve at  $C$  (whose equation you will need to compute).

Measure the incidence and reflection angles to verify that they are equal.

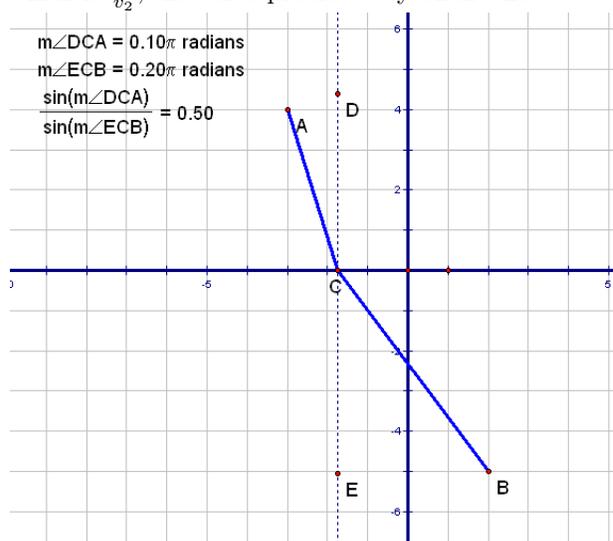
*Solution:* Let  $C = (x, -x^2)$ . We need to find the value of  $x$  that minimizes  $f(x) = |AC| + |BC| = \sqrt{(0-x)^2 + (4+x^2)^2} + \sqrt{(5-x)^2 + (-3+x^2)^2}$ . Using a CAS to compute the derivative and solve  $f'(x) = 0$  numerically, we obtain the solution  $x = 0.7265815885$ .



□

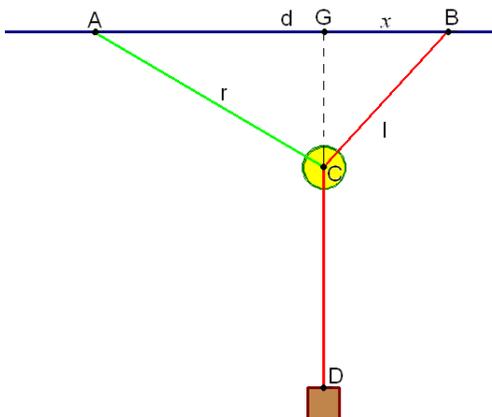
**EXERCISE 10.** Use calculus and a CAS to compute the value of the  $x$  coordinate of the point  $C$  on the  $x$  axis that minimizes the time of travel along the path  $A \rightarrow C \rightarrow B$  with  $A = (-3, 4)$ ,  $B = (2, -5)$ ,  $v_1 = 1$  unit/sec, and  $v_2 = 2$  units/sec. Construct a GS sketch that contains the points  $A$ ,  $B$ , and  $C$ . Calculate the ratio  $\frac{\sin \theta_1}{\sin \theta_2}$ . Is this value the one predicted by Snell's Law?

*Solution:* We need to find the value of  $x$  that minimizes  $\frac{\sqrt{(x+3)^2+(0-4)^2}}{1} + \frac{\sqrt{(x-2)^2+(0+5)^2}}{2}$ . Taking a derivative, setting it equal to 0, and solving, we find the solution  $x = -1.743575120$ . Calculating  $\frac{\sin \theta_1}{\sin \theta_2}$  using GS gives .5, which is  $\frac{v_1}{v_2}$ , the value predicted by Snell's Law.



□

**EXERCISE 11.** Consider L'Hôpital's pulley problem, illustrated below.



In the picture  $AB = d$ ,  $AC = r$ , and  $BC + CD = l$ . Let  $x$  be the horizontal distance between  $G$  and  $B$ .

- (a) Find an expression for  $|GA|$ .

*Solution:*  $GA = d - x$ .

□

- (b) Find an expression for  $|GC|$ .

*Solution:*  $GC = \sqrt{r^2 - (d - x)^2}$ .

□

- (c) Find an expression for  $|BC|$ .

*Solution:*  $BC = \sqrt{x^2 + r^2 - (d - x)^2}$ .

□

- (d) Find an expression for  $|CD|$ .

*Solution:*  $CD = l - \sqrt{x^2 + r^2 - (d - x)^2}$ .

□

- (e) Find an expression for  $|GD|$ .

*Solution:*  $GD = \sqrt{r^2 - (d - x)^2} + l - \sqrt{x^2 + r^2 - (d - x)^2}$ .

□

- (f) According to the principle of minimum potential energy, the system will reach equilibrium when  $|GD|$  is maximized. Use calculus and a CAS to find the value of  $x$  for which  $|GD|$  is a maximum.

*Solution:* Using a CAS to differentiate the expression with respect to  $x$  and setting it equal to 0, we obtain the solutions

$$x = \frac{4d^2 - r^2}{4d} \pm \frac{\sqrt{8d^2 + r^2}|r|}{4d}$$

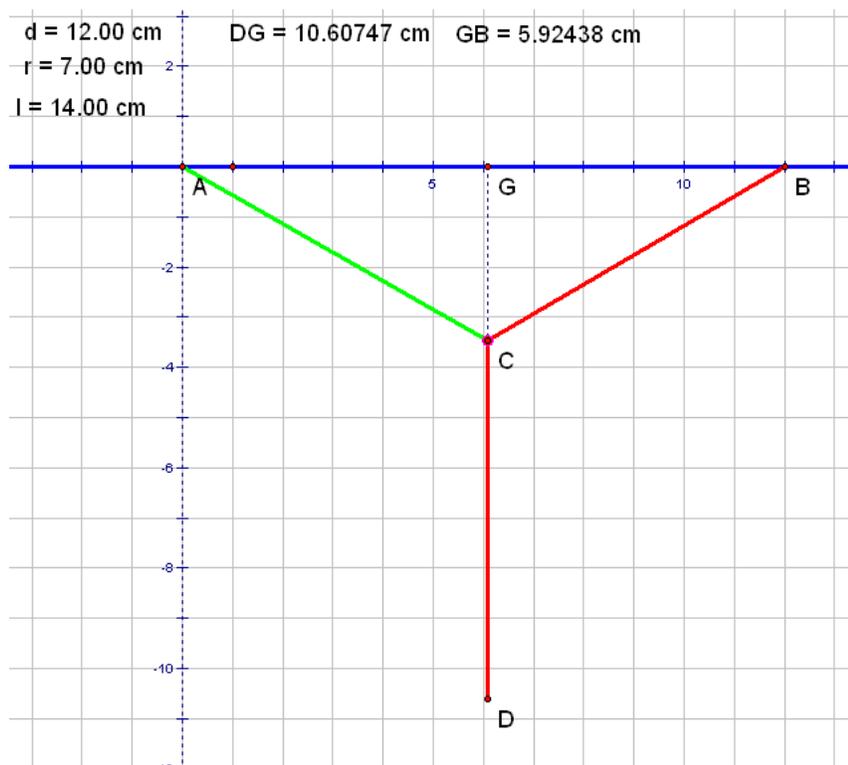
□

- (g) Compute  $x$  when  $d = 12$ ,  $r = 7$ , and  $l = 14$ .

*Solution:* Plugging in  $d = 12$ ,  $r = 7$ , and  $l = 14$  above we obtain  $x = 5.925247326$  or  $x = 16.03308600$  or  $x = 0$ . Since  $x$  has to be between 0 and 12, the only solution that makes sense is  $x = 5.925247326$ .

□

- (h) The pulley problem is illustrated on Page 1 of the GS file Optimization.gsp. Drag  $C$  to locate the equilibrium position where  $GD$  is a maximum. Does  $GB$  agree with your solution above?

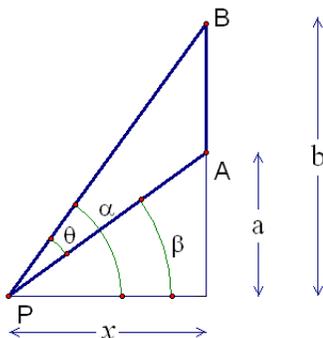


*Solution:*

Yes, the maximum value of  $GD$  is 10.60747, and it occurs when  $x = 5.92438$ , which agrees (to two decimal places) with the answer obtained above.  $\square$

## EXERCISE 12.

- (a) Use calculus and a CAS to find the solution of Regiomontanus' hanging picture problem. You need to find the value of  $x$  that maximizes  $\theta$ .



Since  $f(x) = \tan x$  is an increasing function, it is equivalent to maximize  $\tan \theta = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$ . You should be able to express this last expression all in terms of  $x$ ,  $a$ , and  $b$ . Then use calculus to find the value of  $x$  that maximizes this quantity.

*Solution:* We need to find the value of  $x$  that maximizes  $\tan(\theta) = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x} \cdot \frac{a}{x}}$ . Using a CAS to take a derivative and set it equal to zero, we obtain the solution  $x = \sqrt{ab}$ .  $\square$

- (b) Suppose a six-foot tall adult comes to the museum with his three-foot tall child. They go into a room where a large 12-ft high painting is hung on the wall so that the bottom edge of the painting is 8 feet from the floor.

- How far from the painting should each of them stand in order to maximize their individual viewing angles?

*Solution:* The six-foot tall adult's eyes are two feet below the bottom of the painting. So  $a = 2$  and  $b - a = 12$ . Hence  $b = 14$  and so he should stand  $\sqrt{2 \cdot 14} \approx 5.291502622$  feet away from the painting.

The three-foot tall child's eyes are five feet below the bottom of the painting. So  $a = 5$  and  $b - a = 12$ . Hence  $b = 17$  and so he should stand  $\sqrt{5 \cdot 17} \approx 9.219544457$  feet away from the painting.  $\square$

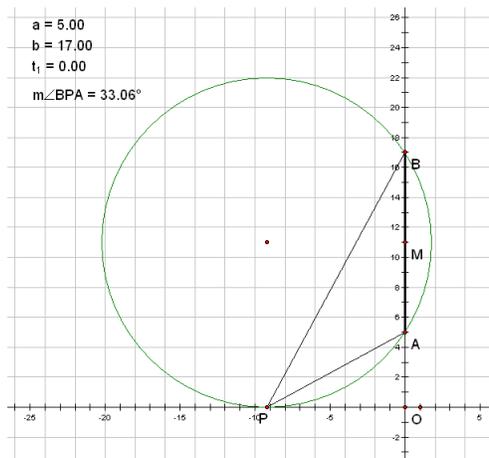
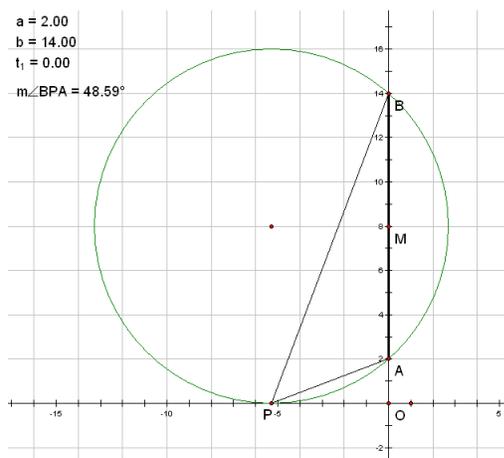
- Compute each of their maximal viewing angles. Express your answer in degrees.

*Solution:* Plugging  $a = 2$ ,  $b = 14$ , and  $x = \sqrt{28}$  into the expression for  $\tan(\theta)$  we get  $\tan(\theta) = 1.133893419$ , so  $\theta = 0.8480620789$  radians. To convert to degrees we multiply by  $\frac{180 \text{ deg}}{\pi \text{ radians}}$ . Thus  $\theta = \frac{0.8480620789 \cdot 180}{\pi} \approx 48.59037788$  degrees.

Plugging  $a = 5$ ,  $b = 17$ , and  $x = \sqrt{85}$  into the expression for  $\tan(\theta)$  we get  $\tan(\theta) = 0.6507913734$ , so  $\theta = 0.5769313452$  radians. To convert to degrees we multiply by  $\frac{180 \text{ deg}}{\pi \text{ radians}}$ . Thus  $\theta = \frac{0.5769313452 \cdot 180}{\pi} \approx 33.05573114$  degrees.  $\square$

- Page 2 of the GS file Optimization.gsp illustrates Regiomontanus' Hanging Picture, with the line segment  $AB$  denoting the height of the picture, and the  $x$  axis representing eye level. Construct a circle through  $A$  and  $B$  that is tangent to the  $x$  axis. (Note that the radius of this circle will have length  $MO$  where  $M$  is the midpoint of  $AB$ .) Verify that the point of tangency  $P$  is the solution to the Regiomontanus' problem by measuring  $\angle APB$  and verifying that it equals the maximal viewing angle you found above for the six-foot adult. Change the values of the parameters to  $a = 5$  and  $b = 17$  to verify that the measure of the resulting angle  $APB$  equals the maximal viewing angle you found above for the three-foot child.

*Solution:*



□

**EXERCISE 13.** Use a CAS to define the following two functions:

$$T(a, b, c, d, u) := \frac{\sqrt{(a-c)^2 + (b-d)^2} \cdot \left( \sqrt{1960(b-d) + u^2} - u \right)}{980(b-d)}$$

$$V(a, b, c, d, u) := \sqrt{1960(b-d) + u^2}$$

$T(a, b, c, d, u)$  computes the time of descent from  $A(a, b)$  to  $B(c, d)$  along a straight-line path, assuming no friction and initial velocity  $u$ .  $V(a, b, c, d, u)$  gives the terminal velocity. We assume distance is measured in centimeters and time in seconds.

- (a) Compute  $T(5, 5, 0, 0, 0)$  to find the time of descent from  $A(5, 5)$  to  $B(0, 0)$  along an inclined plane, assuming no friction and an initial velocity of 0.

*Solution:*  $T(5, 5, 0, 0, 0) = 0.1428571428$  seconds. □

- (b) To compute the time of descent along the polygonal path  $(5, 5) \rightarrow (3, 4) \rightarrow (0, 0)$ , you need to add the time it takes along  $(5, 5) \rightarrow (3, 4)$ , which is  $T(5, 5, 3, 4, 0)$ , and the time it takes along  $(3, 4) \rightarrow (0, 0)$ , which is  $T(3, 4, 0, 0, u)$  where  $u$  is the terminal velocity from the first part of the path, i.e.  $u = V(5, 5, 3, 4, 0)$ . Hence compute

$$T(5, 5, 3, 4, 0) + T(3, 4, 0, 0, V(5, 5, 3, 4, 0))$$

*Solution:*

$$T(5, 5, 3, 4, 0) + T(3, 4, 0, 0, V(5, 5, 3, 4, 0)) = 0.1708150785 \text{ seconds}$$

□

- (c) Suppose we want a polygonal path  $(5, 5) \rightarrow (x, y) \rightarrow (0, 0)$  where  $(x, y)$  lies on the circle  $x^2 + (y-5)^2 = 25$ . Solving for  $y$ , where  $0 \leq y \leq 5$ , we obtain  $y = 5 - \sqrt{25 - x^2}$ , so our point  $(x, y)$  must be of the form  $(x, 5 - \sqrt{25 - x^2})$  in order to lie on the quarter circle. The time of descent for this path then is

$$T(5, 5, x, 5 - \sqrt{25 - x^2}, 0) + T(x, 5 - \sqrt{25 - x^2}, 0, 0, V(5, 5, x, 5 - \sqrt{25 - x^2}, 0))$$

Use a CAS and calculus to find the value of  $x \in (0, 5)$  that minimizes the above quantity. Compare the time of descent along this polygonal path to the time along the straight line path.

*Solution:* Taking a derivative, setting it equal to 0, and solving we find that  $x = 4.550898608$ . The  $y$  value is  $5 - \sqrt{25 - x^2} = 2.928932193$ . So  $C = (4.550898608, 2.928932193)$ . The time of descent along this path is

$$T(5, 5, 4.550898608, 2.928932193, 0) + T(4.550898608, 2.928932193, 0, 0, V(5, 5, 4.550898608, 2.928932193, 0)) = 0.1330475855 \text{ seconds}$$

This is less than 0.1708150785, the time along the path  $(5, 5) \rightarrow (0, 0)$ . □

- (d) Let  $C$  be the point that you found above. Galileo knew that any polygonal path  $(5, 5) \rightarrow C \rightarrow (0, 0)$  where  $C$  is on the circle would take less time than the straight line path  $(5, 5) \rightarrow (0, 0)$ . He then reasoned that if you added another point  $D$  on the circle to form the path  $(5, 5) \rightarrow C \rightarrow D \rightarrow (0, 0)$ , the time of descent for this path would be less than the time for the path  $(5, 5) \rightarrow C \rightarrow (0, 0)$ . Find  $D = (x, 5 - \sqrt{25 - x^2})$ , so that the time of descent for the polygonal path  $(5, 5) \rightarrow C \rightarrow D \rightarrow (0, 0)$  is smallest. What is the total time for the path  $(5, 5) \rightarrow C \rightarrow D \rightarrow (0, 0)$ ?

*Solution:* Let  $v_1 = V(5, 5, 4.550898608, 2.956882759, 0) = 63.71258040$  cm/sec, the terminal velocity along  $(5, 5) \rightarrow (4.550898608, 2.956882759)$ . We need to find  $x$  that minimizes

$$T(4.550898608, 2.956882759, x, 5 - \sqrt{25 - x^2}, v_1) \\ + T(x, 5 - \sqrt{25 - x^2}, 0, 0, V(4.550898608, 2.956882759, x, 5 - \sqrt{25 - x^2}, v_1))$$

Taking a derivative and setting it equal to 0, we find  $x = 2.956882754$ . The corresponding  $y$  coordinate is  $5 - \sqrt{25 - x^2} = 0.9680222744$ , so  $D = (2.956882754, 0.9680222744)$ .

The terminal velocity as the object reaches point  $D$  is

$$v_2 = V(4.550898608, 2.956882759, 2.956882754, 0.9680222744, 63.71258040) = 89.20459321 \text{ cm/sec.}$$

The total time along the path  $(5, 5) \rightarrow C \rightarrow D \rightarrow (0, 0)$  is

$$T(5, 5, 4.550898608, 2.956882759, 0) \\ + T(4.550898608, 2.956882759, 2.956882754, 0.9680222744, 63.71258040) \\ + T(2.956882754, 0.9680222744, 0, 0, 89.20459321) = 0.1324655394 \text{ seconds}$$

□

#### EXERCISE 14.

- (a) Use calculus to find the value of  $x$  so that the time of descent along  $(5, 5) \rightarrow (x, 4) \rightarrow (0, 0)$  is the smallest. Let  $C = (x_1, 4)$  be the point that minimizes the time of descent. Also compute the terminal velocity  $v_1$  on the path  $(5, 5) \rightarrow C$ .

*Solution:* We need to find the value of  $x$  that minimizes  $T(5, 5, x, 4, 0) + T(x, 4, 0, 0, V(5, 5, x, 4, 0))$ . Taking the derivative and setting it equal to zero, the solution is  $x = 4.756587967$ . So  $C = (4.756587967, 4)$ . The terminal velocity as the object reaches point  $C$  is  $V(5, 5, 4.756587967, 4, 0) = 44.27188724$ . □

- (b) Use calculus to find the value of  $x$  so that the time of descent along  $C \rightarrow (x, 3) \rightarrow (0, 0)$  is the smallest, where the initial velocity at  $C$  is  $v_1$ . Let  $D = (x_2, 3)$  be the point that minimizes the time of descent. Also compute the terminal velocity  $v_2$  on the path  $C \rightarrow D$ .

*Solution:* We need to find the value of  $x$  that minimizes  $T(4.756587967, 4, x, 3, 44.27188724) + T(x, 3, 0, 0, V(4.756587967, 4, x, 3, 44.27188724))$ . Taking the derivative and setting it equal to zero, the solution is  $x = 4.123641464$ . So  $D = (4.123641464, 3)$ . The terminal velocity as the object reaches point  $D$  is  $V(4.756587967, 4, 4.123641464, 3, 44.27188724) = 62.60990336$ . □

- (c) Use calculus to find the value of  $x$  that minimizes the time of descent along  $D \rightarrow (x, 2) \rightarrow (0, 0)$  is the smallest, where the initial velocity at  $D$  is  $v_2$ . Let  $E = (x_3, 2)$  be the point that minimizes the time of descent. Also compute the terminal velocity  $v_3$  on the path  $D \rightarrow E$ .

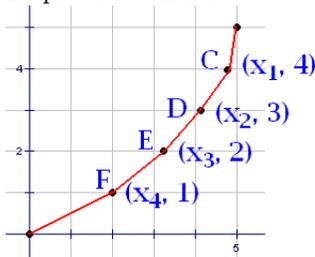
*Solution:* We need to find the value of  $x$  that minimizes  $T(4.123641464, 3, x, 2, 62.60990336) + T(x, 2, 0, 0, V(4.123641464, 3, x, 2, 62.60990336))$ . Taking the derivative and setting it equal to zero, the solution is  $x = 3.213368724$ . So  $E = (3.213368724, 2)$ . The terminal velocity as the object reaches point  $E$  is  $V(4.123641464, 3, 3.213368724, 2, 62.60990336) = 76.68115805$ . □

- (d) Finally, use calculus to find the value of  $x$  that minimizes the time of descent along  $E \rightarrow (x, 1) \rightarrow (0, 0)$ , where the initial velocity at  $E$  is  $v_3$ . Let  $F = (x_4, 1)$  be the point that minimizes the time of descent. Also compute the terminal velocity  $v_4$  on the path  $E \rightarrow F$ .

*Solution:* We need to find the value of  $x$  that minimizes  $T(3.213368724, 2, x, 1, 76.68115805) + T(x, 1, 0, 0, V(3.213368724, 2, x, 1, 76.68115805))$ . Taking the derivative and setting it equal to zero, the solution is  $x = 1.950468697$ . So  $F = (1.950468697, 1)$ . The terminal

velocity as the object reaches point  $F$  is  $V(3.213368724, 2, 1.950468697, 1, 76.68115805) = 88.54377448$ .  $\square$

- (e) Compute the time of descent along the the path  $(5, 5) \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow (0, 0)$ . How does this compare to the time of descent along the polygonal path  $(5, 5) \rightarrow C \rightarrow D \rightarrow (0, 0)$  that you computed in the previous exercise?



*Solution:* The time of descent is

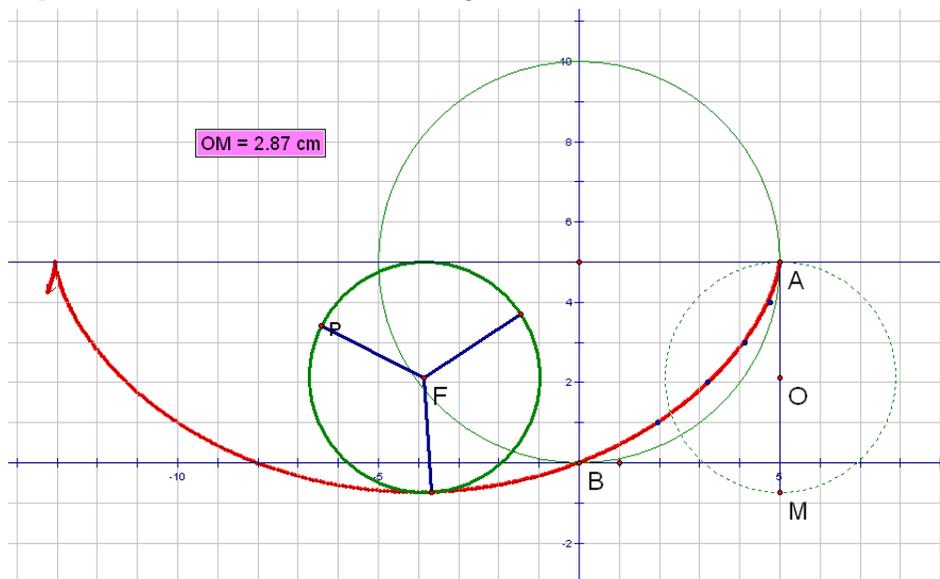
$$\begin{aligned} &T(5, 5, 4.756587967, 4, 0) + T(4.756587967, 4, 4.123641464, 3, 44.27188724) \\ &+ T(4.123641464, 3, 3.213368724, 2, 62.60990336) + T(3.213368724, 2, 1.950468697, 1, 76.68115805) \\ &+ T(1.950468697, 1, 0, 0, 88.54377447) = 0.1309306400 \text{ seconds} \end{aligned}$$

This is slightly less than 0.1324655394, the time along the polygonal path from the previous exercise.  $\square$

- (f) Open Page 3 of the GS file Optimization.gsp. This sketch contains the construction of an (upside-down) cycloid, a curve traced out by a point on the circumference of a circle as it rolls along a straight line. You can generate the cycloid by right-clicking on the center  $F$  of the rolling circle and selecting “Animate Point”.
- (g) Now obtain a graph of the entire curve by selecting  $F$  and  $P$ , and then choosing **Construct**  $\gg$  **Locus**.
- (h) The curve goes through  $A(5, 5)$ , but it doesn't go through  $B(0, 0)$ . You can find a cycloid that goes through  $B$  by adjusting the size of the rolling circle (drag  $M$  up and down to to this). What is the radius of the generating circle that produces a cycloid that goes through both  $A$  and  $B$ ?

*Solution:*  $r \approx 2.87$   $\square$

- (i) It turns out that this cycloid is the answer to the brachistochrone problem (for any points  $A$  and  $B$ , not just  $A = (5, 5)$  and  $B = (0, 0)$ ). Plot the points on the polygonal path that you found above. Your cycloid should be a pretty good fit. You should also construct a circle with center  $(0, 5)$  and radius 5, so you can compare the actual solution to Galileo's guess.



*Solution:*  $\square$