

WHAT IS A GROUP RING?

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1. Introduction. Let K be a field. Suppose we are given some three element set $\{\alpha, \beta, \gamma\}$ and we are asked to form a K -vector space V with this set as a basis. Then certainly we merely take V to be the collection of all formal sums $a \cdot \alpha + b \cdot \beta + c \cdot \gamma$ with $a, b, c \in K$. In the same way if we were originally given four, five or six element sets we would again have no difficulty in performing this construction. After awhile, however, as the sets get larger, the plus sign becomes tedious and at that point we would introduce the Σ notation. In general, if we are given a finite set S , then the K -vector space V with basis S consists of all formal sums $\sum_{\alpha \in S} a_\alpha \cdot \alpha$ with coefficients $a_\alpha \in K$. Finally, there is no real difficulty in letting S become infinite. We merely restrict the sums $\sum a_\alpha \cdot \alpha$ to be finite, by which we mean that only finitely many nonzero coefficients a_α can occur.

Of course, addition in V is given by

$$\left(\sum a_\alpha \cdot \alpha \right) + \left(\sum b_\alpha \cdot \alpha \right) = \sum (a_\alpha + b_\alpha) \cdot \alpha$$

and scalar multiplication is just

$$b \left(\sum a_\alpha \cdot \alpha \right) = \sum (ba_\alpha) \cdot \alpha.$$

Moreover, by identifying $\beta \in S$ with the element $\beta' = \sum b_\alpha \cdot \alpha \in V$, where $b_\beta = 1$ and $b_\alpha = 0$ for $\alpha \neq \beta$ we see that V does indeed have this copy of S as a basis and our original problem is solved.

Now, how do we multiply elements of V ? Certainly the coefficient-by-coefficient multiplication

$$\left(\sum a_\alpha \cdot \alpha \right) \circ \left(\sum b_\alpha \cdot \alpha \right) = \sum (a_\alpha b_\alpha) \cdot \alpha$$

is exceedingly uninteresting and other than that no natural choice seems to exist. So we are stuck. But suppose finally that we are told that S is not just any set, but rather that S is in fact a multiplicative group. We would then have a natural multiplication for the basis elements and by way of the distributive law this could then be extended to all of V .

Let us now start again. Let K be a field and let G be a multiplicative group, not necessarily finite. Then the **group ring** $K[G]$ is a K -vector space with basis G and with multiplication defined distributively using the given multiplication of G . In other words, for the latter we have

$$\left(\sum_{x \in G} a_x \cdot x \right) \left(\sum_{y \in G} b_y \cdot y \right) = \sum_{x, y \in G} (a_x b_y) \cdot (xy) = \sum_{z \in G} c_z \cdot z,$$

where

$$c_z = \sum_{xy=z} a_x b_y = \sum_{x \in G} a_x b_{x^{-1}z}.$$

Certainly the associative law in G guarantees the associativity of multiplication in $K[G]$ so $K[G]$ is a ring and in fact a K -algebra. It is clear that these easily defined group rings offer rather attractive objects of study. Furthermore, as the name implies, this study is a meeting place for two essentially different disciplines and indeed the results are frequently a rather nice blending of group theory and ring theory.

Suppose for a moment that G is finite so that $K[G]$ is a finite dimensional K -algebra. Since the study of finite dimensional K -algebras (especially semisimple ones over algebraically closed fields) is in far better shape than the study of finite groups, the group ring $K[G]$ has historically been used as a tool of group theory. This is of course what the ordinary and modular character theory is all about (see [2] for example). On the other hand, if G is infinite then neither the group theory nor the ring theory is particularly advanced and what becomes interesting here is the interplay between the two. Our main concern in this article will be with infinite groups and in the following we shall discuss and prove a few

selected results. These results, however, are by no means representative of the entire scope of the subject, but rather were chosen because their proofs are both elegant and elementary. Few references will be given here, but the interested reader is invited to consult the book [5] or the more recent surveys, [4] and [6].

2. Zero divisors. Before we go any further, two observations are in order. First, once we identify the elements of G with a basis of $K[G]$, then the formal sums and products in $\sum a_x \cdot x$ are actually ordinary sums and products. In particular, we shall drop the dot in this notation. Second, if H is a subgroup of G then, since H is a subset of the basis G , its K -linear span is clearly just $K[H]$. Thus $K[H]$ is embedded naturally in $K[G]$.

Now let x be a nonidentity element of G so that G contains the cyclic group $\langle x \rangle$. Then $K[G]$ contains $K[\langle x \rangle]$ and we briefly consider the latter group ring. Suppose first that x has finite order $n > 1$. Then $1, x, \dots, x^{n-1}$ are distinct powers of x and the equation

$$(1 - x)(1 + x + \dots + x^{n-1}) = 1 - x^n = 0$$

shows that $K[\langle x \rangle]$, and hence $K[G]$, has a proper divisor of zero. On the other hand, if x has infinite order then all powers of x are distinct and $K[\langle x \rangle]$ consists of all finite sums of the form $\sum_{i=-\infty}^{\infty} a_i x^i$. Thus this group ring looks something like the polynomial ring $K[x]$ and indeed every element of $K[\langle x \rangle]$ is just a polynomial in x divided by some sufficiently high power of x . Thus $K[\langle x \rangle]$ is contained in the rational function field $K(x)$ and is therefore an integral domain.

Now what we have shown above is the following. If G has a nonidentity element of finite order, a torsion element, then $K[G]$ has a nontrivial divisor of zero, but if G has no nonidentity element of finite order, then there are at least no obvious zero divisors. Because of this, and with frankly very little additional supporting evidence, it was conjectured that G is torsion free if and only if $K[G]$ has no zero divisors. Remarkably this conjecture has held up for over twenty-five years.

We still know very little about this problem. In fact all we know is that the conjecture is true for some rather simple classes of groups as, for example, free groups, or abelian groups. In the latter case the proof is even quite easy. Thus, suppose that G is torsion free abelian, and let α, β be elements of $K[G]$ with $\alpha\beta = 0$. Then, clearly, α and β belong to $K[H]$ for some finitely generated subgroup $H \subseteq G$ and, by the fundamental theorem of abelian groups, H is just the direct product of the infinite cyclic groups $\langle x_1 \rangle, \langle x_2 \rangle, \dots$ and $\langle x_r \rangle$. It is then quite easy to see that $K[H]$ is contained naturally between the polynomial ring $K[x_1, x_2, \dots, x_r]$ and the rational function field $K(x_1, x_2, \dots, x_r)$. Indeed, $K[H]$ is just the set of all elements in $K(x_1, x_2, \dots, x_r)$ which can be written as a polynomial in x_1, x_2, \dots, x_r divided by some sufficiently high power of $(x_1 x_2 \dots x_r)$. Thus $K[H]$ is certainly an integral domain and hence, clearly, either $\alpha = 0$ or $\beta = 0$. Actually the best result to date on this conjecture concerns supersolvable groups. Here for the first time nontrivial ring theory comes into play but the proof is unfortunately too complicated to give here.

On a more positive note, there is a variant of the zero divisor problem which we can handle effectively. A ring R is said to be **prime** if $\alpha R \beta = 0$ for $\alpha, \beta \in R$ implies that $\alpha = 0$ or $\beta = 0$. Clearly this agrees with the usual definition in the commutative case. In addition, this concept makes more sense and is more important from a ring-theoretic point of view than the very much more stringent condition of no zero divisors. For example, the matrix ring K_n is always prime, even though it does have zero divisors for $n \geq 2$. The main result of interest here is as follows.

THEOREM I. *The group ring $K[G]$ is prime if and only if G has no nonidentity finite normal subgroup.*

The proof in one direction is quite easy. Suppose H is a nonidentity finite normal subgroup of G and set $\alpha = \sum_{h \in H} h$, the sum of the finitely many elements of H . If $h \in H$ then $hH = H$ so $h\alpha = \alpha$ and thus

$$\alpha^2 = \sum_{h \in H} h\alpha = |H|\alpha.$$

In particular, if $\beta = |H|1 - \alpha$ then $\alpha\beta = 0$. Furthermore, since H is normal in G , we have for all $x \in G$, $Hx = xH$, and hence $\alpha x = x\alpha$. Therefore α commutes with a basis of $K[G]$, so it is central and we have

$$\alpha K[G]\beta = K[G]\alpha\beta = 0.$$

Finally, since α and β are clearly not zero we see that $K[G]$ is not prime.

The converse direction is much more difficult and as a starter we must ask ourselves how we are going to find a finite normal subgroup in G . Observe that if H is such a subgroup and if $h \in H$, then certainly h has finite order, and furthermore, for all $x \in G$ we have

$$h^x = x^{-1}hx \in x^{-1}Hx = H.$$

Thus all G -conjugates of h are contained in H and so certainly there are only finitely many distinct ones. This, therefore, leads us to define two interesting subsets of G , namely,

$$\Delta(G) = \{x \in G \mid x \text{ has only finitely many conjugates in } G\}$$

and

$$\Delta^+(G) = \{x \in G \mid x \text{ has only finitely many conjugates in } G \text{ and } x \text{ has finite order}\}.$$

At this point a certain amount of group theory obviously comes into play. While the proofs are by no means difficult it does seem inappropriate to offer them here. Therefore we just tabulate the necessary facts below. See [5, Lemma 19.3] for details.

LEMMA 1. *Let G be a group and let $\Delta(G)$ and $\Delta^+(G)$ be defined as above. Then*

- (i) $\Delta(G)$ and $\Delta^+(G)$ are normal subgroups of G .
- (ii) $\Delta(G) \supseteq \Delta^+(G)$ and the quotient group $\Delta(G)/\Delta^+(G)$ is torsion free abelian.
- (iii) $\Delta^+(G) \neq \langle 1 \rangle$ if and only if G has a nonidentity finite normal subgroup.

Part (iii) above certainly seems to answer our first question. Now observe that if x is any element of G , then the G -conjugates of x are in a natural one-to-one correspondence with the right cosets of $\mathbb{C}_G(x)$, the centralizer of x . Thus $x \in \Delta(G)$ if and only if $[G : \mathbb{C}_G(x)] < \infty$ and we shall need the following two basic properties of subgroups of finite index ([5, Lemmas 1.1 and 1.2]).

LEMMA 2. *Let G be a group and let H_1, H_2, \dots, H_n be a finite collection of subgroups of G .*

- (i) *If $[G : H_i]$ is finite for all i , then $[G : \cap H_i]$ is finite.*
- (ii) *If G is the set theoretic union $G = \bigcup_i H_i g_i$ of finitely many right cosets of the subgroups H_i , then for some i we have $[G : H_i]$ finite.*

We can now proceed with the remainder of the proof of Theorem I. If $\alpha = \sum a_x x \in K[G]$, let us define the **support of α** , $\text{Supp } \alpha$, to be the set of all group elements which occur with nonzero coefficient in this expression for α . Thus $\text{Supp } \alpha$ is a finite subset of G which is empty precisely when $\alpha = 0$. Now suppose that $K[G]$ is not prime and choose nonzero group ring elements α and β with $\alpha K[G]\beta = 0$. If $x \in \text{Supp } \alpha$ and $y \in \text{Supp } \beta$ then certainly $(x^{-1}\alpha)K[G](\beta y^{-1}) = 0$, and furthermore, $1 \in \text{Supp } x^{-1}\alpha$, $1 \in \text{Supp } \beta y^{-1}$. Thus without loss of generality, we may assume that α and β have 1 in their supports.

Let us now write $\alpha = \alpha_0 + \alpha_1$ and $\beta = \beta_0 + \beta_1$ where $\text{Supp } \alpha_0, \text{Supp } \beta_0 \subseteq \Delta(G)$ and $\text{Supp } \alpha_1, \text{Supp } \beta_1 \subseteq G - \Delta(G)$. In other words, we have split α and β into two partial sums, the first one containing the group elements in $\Delta(G)$ and the second one containing the rest. Since 1 is contained in the supports of α and β we see that α_0 and β_0 are nonzero elements of $K[\Delta(G)]$. Our goal here is to prove that $\alpha_0\beta_0 = 0$. Once this is done the theorem will follow easily. Let us suppose by way of contradiction that $\alpha_0\beta_0 \neq 0$. Then clearly $\alpha_0\beta = \alpha_0\beta_0 + \alpha_0\beta_1$ is not zero since $\text{Supp } \alpha_0\beta_0 \subseteq \Delta(G)$ and

$\text{Supp } \alpha_0 \beta_1 \subseteq G - \Delta(G)$ so there can be no cancellation between these two summands. Thus we can choose $z \in G$ to be a fixed element in $\text{Supp } \alpha_0 \beta$.

We observe now that if $u \in \text{Supp } \alpha_0$ then $[G : \mathbb{C}_G(u)]$ is finite and thus, by Lemma 2 (i), $H = \bigcap_{u \in \text{Supp } \alpha_0} \mathbb{C}_G(u)$ has finite index in G . Moreover, if $h \in H$, then h centralizes all elements in $\text{Supp } \alpha_0$ and hence h centralizes α_0 . Let us now write $\text{Supp } \alpha_1 = \{x_1, x_2, \dots, x_r\}$ and $\text{Supp } \beta = \{y_1, y_2, \dots, y_s\}$. Furthermore, if x_i is conjugate to zy_j^{-1} in G for some i, j , then we choose some fixed $g_{ij} \in G$ with $g_{ij}^{-1} x_i g_{ij} = zy_j^{-1}$.

Now let $h \in H$ and recall that $\alpha K[G] \beta = 0$. Then since $\alpha h \beta = 0$ and h centralizes α_0 , we have

$$0 = h^{-1} \alpha h \beta = h^{-1} (\alpha_0 + \alpha_1) h \cdot \beta = \alpha_0 \beta + h^{-1} \alpha_1 h \beta.$$

Thus z occurs in the support of $h^{-1} \alpha_1 h \beta = -\alpha_0 \beta$ so there must exist i, j with $z = h^{-1} x_i h y_j$ or in other words $h^{-1} x_i h = zy_j^{-1}$. But then this says that x_i and zy_j^{-1} are conjugate in G , so by definition of g_{ij} we have

$$h^{-1} x_i h = zy_j^{-1} = g_{ij}^{-1} x_i g_{ij},$$

and thus hg_{ij}^{-1} centralizes x_i . We have therefore shown that for each $h \in H$ there exists an appropriate i, j with $h \in \mathbb{C}_G(x_i)g_{ij}$ and hence

$$H \subseteq \bigcup_{i,j} \mathbb{C}_G(x_i)g_{ij}.$$

Now $[G : H]$ is finite, so $G = \bigcup_k Hw_k$ is a finite union of right cosets of H and we conclude from the above that

$$G = \bigcup_{i,j,k} \mathbb{C}_G(x_i)g_{ij}w_k.$$

We have therefore shown that G can be written as a finite set theoretic union of cosets of the subgroups $\mathbb{C}_G(x_i)$ and hence we deduce from Lemma 2 (ii) that for some i , $[G : \mathbb{C}_G(x_i)]$ is finite. But this says that $x_i \in \Delta(G)$ and this is the required contradiction, since by definition $x_i \in \text{Supp } \alpha_1 \subseteq G - \Delta(G)$. Thus $\alpha_0 \beta_0 = 0$.

We can now complete the proof quite quickly and easily. Since α_0 and β_0 are non zero elements of $K[\Delta(G)]$ with $\alpha_0 \beta_0 = 0$ we see that $K[\Delta(G)]$ has nontrivial zero divisors. Hence $\Delta(G)$ cannot possibly be torsion free abelian. On the other hand, according to Lemma 1 (ii), $\Delta(G)/\Delta^+(G)$ is torsion free abelian so we must have $\Delta^+(G) \neq \langle 1 \rangle$. Thus finally we deduce from Lemma 1 (iii) that G has a nonidentity finite normal subgroup and the theorem is proved.

In conclusion, we remark that Theorem I has an amusing application to the zero divisor problem. Namely, we can show that if G is a torsion free group then $K[G]$ has nontrivial zero divisors if and only if it has nonzero elements of square zero. Of course if $\alpha \in K[G]$, $\alpha \neq 0$ with $\alpha^2 = 0$, then certainly $K[G]$ has zero divisors, so this direction is really trivial. In the other direction let α and β be nonzero elements of $K[G]$ with $\alpha\beta = 0$. Since G is torsion free, Theorem I implies immediately that $K[G]$ is prime so we have $\beta K[G] \alpha \neq 0$. But

$$\beta K[G] \alpha \cdot \beta K[G] \alpha = \beta K[G] (\alpha\beta) K[G] \alpha = 0$$

since $\alpha\beta = 0$ and hence we see that every element of $\beta K[G] \alpha$ has square zero.

3. Idempotents. Let us return again to the group ring of a finite group and consider its regular representation. That is, we view $V = K[G]$ as a K -vector space on which $K[G]$ acts as linear transformations by right multiplication. In particular, since V is finite dimensional here, each choice of a basis for V gives rise to a certain matrix representation for $K[G]$. More precisely, for each such

basis we obtain an appropriate faithful homomorphism $\tau : K[G] \rightarrow K_n$ from $K[G]$ into K_n , the ring of $n \times n$ matrices over K . Clearly $n = \dim V = |G|$.

Suppose first that τ corresponds to the natural basis for V , namely G itself. Then for each $x \in G$ right multiplication by x merely permutes the basis elements. Thus $\tau(x)$ is a permutation matrix, that is a matrix of 0's and 1's having precisely one 1 in each row and column. Moreover, if $x \neq 1$, then $gx \neq g$ for all $g \in G$ so $\tau(x)$ has only 0 entries on its main diagonal. Hence clearly $\text{trace } \tau(x) = 0$. On the other hand, $\tau(1)$ is the identity matrix, so $\text{trace } \tau(1) = n = |G|$. Now matrix trace maps are K -linear so for $\alpha = \sum a_x x \in K[G]$ we obtain finally

$$\text{trace } \tau(\alpha) = \sum_x a_x \cdot \text{trace } \tau(x) = a_1 |G|.$$

In other words, the trace of $\tau(\alpha)$ is just a fixed scalar multiple of a_1 , the identity coefficient of α .

We are therefore led, for arbitrary groups G , to define a map $\text{tr} : K[G] \rightarrow K$, called **the trace**, by

$$\text{tr} \left(\sum_x a_x \cdot x \right) = a_1.$$

Moreover, it seems reasonable to expect that tr should have certain trace-like behavior and that it should prove to be an interesting object for study. Indeed, we observe immediately that tr is a K -linear functional on $K[G]$, and furthermore, that for $\alpha = \sum a_x x$, $\beta = \sum b_y y$ we have

$$\text{tr } \alpha\beta = \sum_{xy=1} a_x b_y$$

and this is symmetric in α and β since $xy = 1$ if and only if $yx = 1$. Thus $\text{tr } \alpha\beta = \text{tr } \beta\alpha$.

Now one of the most interesting questions about tr concerns idempotents $e \in K[G]$ and the values taken on by $\text{tr } e$. To see what we might expect these values to be, let us again assume that G is finite. Since the trace of a linear transformation is independent of the choice of basis, we compute

$$|G| \text{tr } e = \text{trace } \tau(e)$$

by taking a basis more appropriate for e . Namely we write V as the vector space direct sum $V = Ve + V(1-e)$ and then we choose as a basis for V the union of ones for Ve and for $V(1-e)$. Since e acts like one on the first set and like zero on the second, we conclude therefore that

$$|G| \text{tr } e = \text{trace } \tau(e) = \dim Ve.$$

Thus if we are able to divide by $|G|$ in K (that is, if the characteristic of K does not divide $|G|$) then

$$\text{tr } e = (\dim Ve) / |G|.$$

In other words, we see that $\text{tr } e$ is contained in the prime subfield of K , i.e., the rationals Q if $\text{char } K = 0$ or the Galois field $\text{GF}(p)$ if $\text{char } K = p$. Furthermore, in characteristic 0 since $0 \leq \dim Ve \leq |G|$ we have $0 \leq \text{tr } e \leq 1$.

Now it turns out that these two properties of $\text{tr } e$ are indeed true in general, but we should observe that there is a basic distinction between them. The fact that $\text{tr } e$ is contained in the prime subfield is clearly an algebraic property while the inequality $0 \leq \text{tr } e \leq 1$ is in some sense analytic in nature. With this basic dichotomy in mind, we now proceed to consider some proofs.

THEOREM II. *Let $e \in K[G]$ be an idempotent. Then $\text{tr } e$ is contained in the prime subfield of K .*

We first consider fields K of characteristic $p > 0$, and as it turns out, the property of e we use here is $e^p = e$. The reason for this is that in algebras over fields of characteristic p the p th power map is fairly well behaved. Indeed the identity $(a+b)^p = a^p + b^p$ always holds in commutative K -algebras and for noncommutative ones an appropriate generalization exists. To be more precise let A be a

K -algebra and define $[A, A]$, the commutator subspace of A , to be the subspace generated by all Lie products $[a, b] = ab - ba$ with $a, b \in A$. Then the result we are alluding to is the following and a proof can be found in [5, Lemma 3.4].

LEMMA 3. *Let A be an algebra over a field K of characteristic $p > 0$. If $a_1, a_2, \dots, a_m \in A$ and if $n > 0$ is a given integer then there exists an elements $b \in [A, A]$ with*

$$(a_1 + a_2 + \dots + a_m)^{p^n} = a_1^{p^n} + a_2^{p^n} + \dots + a_m^{p^n} + b.$$

With this fact we can now prove the first part of Theorem II. Let $e = \sum a_x x$ be an idempotent in $K[G]$ and let S denote the subset of $\text{Supp } e$ consisting of all those elements of order a power of p , the characteristic of K . Then since S is finite, there exists an appropriate p th power, say p^s , with $x^{p^s} = 1$ for all $x \in S$. Now let n be any integer with $n \geq s$ and we apply Lemma 3 to $e^{p^n} = e$. Thus there exists an element γ in the commutator subspace of $K[G]$ with

$$e = e^{p^n} = \sum (a_x)^{p^n} x^{p^n} + \gamma,$$

and we proceed to compute the traces of both sides of this equation. Observe first that $\text{tr } \gamma = 0$ since for any $\alpha, \beta \in K[G]$ we have $\text{tr } \alpha\beta = \text{tr } \beta\alpha$ and hence $\text{tr } [\alpha, \beta] = 0$. Also since $n \geq s$ it follows that for any $x \in \text{Supp } e$ we have $x^{p^n} = 1$ if and only if $x \in S$. Thus clearly

$$\text{tr } e = \sum_{x \in S} (a_x)^{p^n} = \left(\sum_{x \in S} a_x \right)^{p^n}.$$

Now this equation holds for all integers $n \geq s$ and in particular if we take $n = s$ and $s + 1$ we obtain

$$(\text{tr } e)^p = \left(\sum_{x \in S} a_x \right)^{p^s p} = \left(\sum_{x \in S} a_x \right)^{p^{s+1}} = \text{tr } e.$$

Therefore $\text{tr } e$ is an element $k \in K$ which satisfies $k^p = k$ and since all such elements k are contained in $\text{GF}(p)$, the theorem is proved in characteristic $p > 0$.

We now proceed to consider the characteristic 0 case and here we will need the following (see [7, Assertions IV, V, VI]).

LEMMA 4. *Let $A = \mathbb{Z}[a_1, a_2, \dots, a_r]$ be an integral domain in characteristic 0 which is finitely generated as a ring over the integers \mathbb{Z} and let $b \in A$ be an element not contained in the rational numbers \mathbb{Q} . Then there exists a maximal ideal M of A such that $F = A/M$ is a field of characteristic $p > 0$ for some prime p and such that the image of b in F is not contained in $\text{GF}(p)$.*

We remark that this fact is an easy consequence of the Extension Theorem for Places if b is transcendental over \mathbb{Q} . But if b is assumed algebraic, then its possible images in the fields A/M are greatly restricted. Fortunately in this case we can apply the Frobenius Density Theorem, a result from algebraic number theory, to prove the lemma.

Now let K have characteristic 0 and let $e = \sum a_x x$ be an idempotent in $K[G]$. If $A = \mathbb{Z}[a_x \mid x \in \text{Supp } e]$, then A is clearly an integral domain in characteristic 0 and A is finitely generated as a ring over the integers \mathbb{Z} . Furthermore, e is an idempotent in $A[G]$, where the latter is the subring of $K[G]$ consisting of all elements with coefficients in A . Now let M be any maximal ideal of A with $F = A/M$ a field of characteristic $p > 0$. Then under the natural homomorphism

$$A[G] \rightarrow (A/M)[G] = F[G]$$

the image of e is an idempotent and hence by the first part of Theorem II the image of $\text{tr } e$ is contained in $\text{GF}(p)$. The second part of Theorem II now follows immediately from Lemma 4 with $b = \text{tr } e$.

Thus we have proved the first conjectured property of $\text{tr } e$, namely that it is always contained in the prime subfield of K and it remains to consider the second property, namely that $0 \leq \text{tr } e \leq 1$ if K has

characteristic 0. Here the proof is much more technical in nature, it is analytic rather than algebraic, and we shall present some basic reductions and then indicate why the proof should work.

Let K be a field of characteristic 0 and let $e = \sum a_x x$ be an idempotent in $K[G]$. If $F = Q(a_x \mid x \in \text{Supp } e)$ then F is a finitely generated field extension of Q and e is an idempotent in $F[G]$. Furthermore, F can be embedded in the complex numbers C and hence again e is an idempotent in $C[G]$. Thus, as a first reduction, we may clearly assume that $K = C$ is the field of complex numbers. A second minor observation is that we need only show that $\text{tr } e \geq 0$. The reason for this is that if e is an idempotent, then so is $1 - e$ and then $1 - \text{tr } e = \text{tr}(1 - e) \geq 0$ yields $\text{tr } e \leq 1$.

Let us now consider the complex group ring $C[G]$. As is to be expected when one combines the structure of group rings with the richness of the complex numbers, many nice additional properties emerge. For example, if $\alpha = \sum a_x x$ and $\beta = \sum b_x x$ are elements of $C[G]$ we set

$$(\alpha, \beta) = \sum_x a_x \bar{b}_x$$

and

$$\|\alpha\| = (\alpha, \alpha)^{1/2} = \left(\sum_x |a_x|^2 \right)^{1/2}$$

where \bar{a} is the complex conjugate of a and $|a|$ is its absolute value. Then clearly $(\ , \)$ defines a Hermitian inner product on $C[G]$ with the group elements as an orthonormal basis and with $\|\ \|$ the usual associated norm. Furthermore let α^* be given by

$$\alpha^* = \sum_x \bar{a}_x x^{-1}.$$

Then the map $*$ is easily seen to satisfy

$$(\alpha + \beta)^* = \alpha^* + \beta^*, (\alpha\beta)^* = \beta^* \alpha^*, \alpha^{**} = \alpha,$$

so that $*$ is a ring antiautomorphism of $C[G]$ of order 2, that is, an involution. Now observe that

$$(\alpha, \beta) = \text{tr } \alpha\beta^* = \text{tr } \beta^* \alpha$$

and hence we deduce easily that if γ is a third element of $C[G]$, then

$$(\alpha, \beta\gamma) = (\alpha\gamma^*, \beta) = (\beta^* \alpha, \gamma).$$

In other words, $*$ is the adjoint map with respect to this inner product for both right and left multiplication.

We can now use the above machinery to obtain an alternate proof of the assertion $\text{tr } e \geq 0$ at least when G is finite. Let $I = eC[G]$ be the right ideal of $C[G]$ generated by the idempotent e and let I^\perp be its orthogonal complement. Then since $C[G]$ is a finite dimensional vector space, we know that $I + I^\perp = C[G]$ is a direct sum decomposition. But I^\perp is not just a subspace of $C[G]$, it is in fact also a right ideal. To see this let $\alpha \in I$, $\beta \in I^\perp$ and $\gamma \in C[G]$. Then since I is a right ideal $\alpha\gamma^* \in I$ and hence

$$(\alpha, \beta\gamma) = (\alpha\gamma^*, \beta) = 0.$$

Thus $\beta\gamma$ is orthogonal to all $\alpha \in I$ and we conclude that $\beta\gamma \in I^\perp$.

We have therefore shown that $I + I^\perp = C[G]$ is a decomposition of $C[G]$ as a direct sum of two right ideals and we let $f + f^\perp = 1$ be the corresponding decomposition of 1. As is well known, f and f^\perp are then both idempotents with $I = fC[G]$ and $I^\perp = f^\perp C[G]$. Now f is orthogonal to $f^\perp C[G] = (1 - f)C[G]$ so for all $\alpha \in C[G]$ we have

$$0 = (f, (1 - f)\alpha) = ((1 - f)^* f, \alpha)$$

and hence $(1-f)^*f \in C[G]^\perp = 0$. Thus $f = f^*f$ so

$$f^* = (f^*f)^* = f^*f = f$$

and we see that f is a self-adjoint idempotent, a projection. It is now an easy matter to complete the proof. Since $eC[G] = I = fC[G]$ both e and f are left identities for this ideal and hence we have

$$\operatorname{tr} e = \operatorname{tr} fe = \operatorname{tr} ef = \operatorname{tr} f.$$

Furthermore $f = f^*f$ so

$$\operatorname{tr} e = \operatorname{tr} f = \operatorname{tr} f^*f = \|f\|^2 \geq 0$$

and the result follows.

It is amusing to observe that this proof yields only $\operatorname{tr} e \geq 0$ and not $\operatorname{tr} e \in Q$, but it does have the redeeming virtue of being extendable to infinite groups. Of course the finiteness of G was used crucially here. In fact it was used to deduce that $I + I^\perp = C[G]$ since such decompositions are not in general true for infinite dimensional inner product spaces. But the proof really rests upon the auxiliary element f and there is an alternate characterization of this element which we can use. Let $\alpha \in I$ and consider the distance between α and $1 \in C[G]$. Then by definition

$$d(\alpha, 1)^2 = \|\alpha - 1\|^2 = (\alpha - 1, \alpha - 1)$$

and since $1 = f + f^\perp$ and $(\alpha - f, f^\perp) = 0$ we have

$$d(\alpha, 1)^2 = (\alpha - f - f^\perp, \alpha - f - f^\perp) = \|\alpha - f\|^2 + \|f^\perp\|^2.$$

Thus $d(\alpha, 1) \geq \|f^\perp\|$ and equality occurs if and only if $\alpha = f$. In other words, f is the unique element of I which is closest to 1.

Now let G be an arbitrary group, let e be an idempotent in $C[G]$ and set $I = eC[G]$. Then we define the distance from I to 1 to be

$$d = \inf_{\alpha \in I} d(\alpha, 1) = \inf_{\alpha \in I} \|\alpha - 1\|.$$

Since $C[G]$ is no longer complete if G is infinite, there is no reason to believe that some element of I exists which is closest to 1. But there certainly exists a sequence of elements of I whose corresponding distances approach d . As it turns out, it is convenient to choose a sequence $f_1, f_2, \dots, f_n, \dots$ of elements of I with

$$d^2 \leq \|f_n - 1\|^2 < d^2 + 1/n^4.$$

Then this sequence plays the role of the auxiliary element f of the finite case and indeed the final conclusion mirrors the original formula $\operatorname{tr} e = \|f\|^2$ obtained above. Namely after a certain amount of work with inequalities and approximations, which we will not give here (see [5, Section 22] for details), we deduce finally that

$$\operatorname{tr} e = \lim_{n \rightarrow \infty} \|f_n\|^2 \geq 0.$$

This completes the proof.

4. Semisimplicity. We close with one final problem, the semisimplicity problem. The amusing thing about this one is that while it has been worked on for over twenty-five years, it is only recently that a viable conjecture has been formulated. In this section we shall discuss the conjecture along with some of the early work in characteristic 0.

We first offer some definitions. If R is a ring, then an R -module V is an additive abelian group which admits right multiplication by the elements of R . More precisely, we are given a ring

homomorphism $R \rightarrow \text{End } V$ from R to the endomorphism ring of V and by way of this map there is a natural action of R on V which we denote by right multiplication. Thus in essence, V is an R -vector space. Furthermore we say that V is an irreducible R -module if there are no R -submodules of V other than 0 or V itself. For example, if $R = F$ is a field then the irreducible F -modules are precisely the one-dimensional F -vector spaces.

Now what we would like to do is study R in terms of its irreducible modules but there is a natural obstruction here. It is quite possible that we cannot even tell the elements of R apart in this manner, that is, there could exist two distinct elements $r, s \in R$ such that for all irreducible R -modules V and for all $v \in V$ we have $vr = vs$. Of course if the above occurs then $v(r - s) = 0$ and hence we cannot distinguish the element $r - s$ from 0. Thus what is of interest here is

$$JR = \{r \in R \mid Vr = 0 \text{ for all irreducible } R\text{-modules } V\},$$

the Jacobson radical of R . This is easily seen to be a two sided ideal of R which can be alternately characterized as follows (see [3, Theorem 1. 2. 3]).

LEMMA 5. *Let R be a ring with 1. Then*

$$JR = \{r \in R \mid 1 - rs \text{ is invertible for all } s \in R\}.$$

Now a ring is said to be **semisimple** if its Jacobson radical is zero and this then leads to two problems in group rings. The first is to characterize those fields K and groups G with $K[G]$ semisimple and then the second, more ambitious problem, is to characterize in general the Jacobson radical $JK[G]$. The initial work here concerned fields of characteristic 0 since for G finite it was a classical fact that $K[G]$ is semisimple. Presumably $K[G]$ is always semisimple in characteristic 0 and the first result here on infinite groups concerned the field C of complex numbers.

THEOREM III. *For all groups G , $C[G]$ is semisimple.*

For our proof, which closely mirrors the original, we revert to the notation of the previous section and we furthermore introduce an auxiliary norm on $C[G]$ by defining $|\alpha| = \sum |a_x|$ if $\alpha = \sum a_x x$. Clearly $|\alpha + \beta| \leq |\alpha| + |\beta|$ and $|\alpha\beta| \leq |\alpha| \cdot |\beta|$. Now let α be a fixed element of $JC[G]$. Then for all complex numbers ζ , $1 - \zeta\alpha$ is invertible by Lemma 5 and we consider the map

$$f(\zeta) = \text{tr}(1 - \zeta\alpha)^{-1}.$$

This is a complex function of the complex variable ζ and we shall show that f is in fact an entire function and we will find its Taylor series about the origin.

For convenience we set $g(\zeta) = (1 - \zeta\alpha)^{-1}$ so that $f(\zeta) = \text{tr } g(\zeta)$. Furthermore, it is clear that all $g(\zeta) \in C[G]$ commute and hence we have the basic identity

$$\begin{aligned} g(\zeta) - g(\eta) &= (1 - \zeta\alpha)^{-1} - (1 - \eta\alpha)^{-1} \\ &= [(1 - \eta\alpha) - (1 - \zeta\alpha)](1 - \zeta\alpha)^{-1}(1 - \eta\alpha)^{-1} \\ &= (\zeta - \eta)\alpha g(\zeta)g(\eta). \end{aligned}$$

We first show that $|g(\eta)|$ is bounded in a neighborhood of ζ . Now by the above $g(\eta) = g(\zeta) - (\zeta - \eta)\alpha g(\zeta)g(\eta)$ so $|g(\eta)| \leq |g(\zeta)| + |\zeta - \eta| |\alpha g(\zeta)| |g(\eta)|$ and hence

$$|g(\eta)| \cdot \{1 - |\zeta - \eta| |\alpha g(\zeta)|\} \leq |g(\zeta)|.$$

In particular, if we choose η sufficiently close to ζ then we can make the factor $\{\dots\}$ larger than $1/2$ and thus we deduce that $\eta \sim \zeta$ implies $|g(\eta)| \leq 2|g(\zeta)|$.

Next we show that $f(\zeta)$ is an entire function. To do this we first plug the above formula for $g(\eta)$ into the right hand side of the basic identity to obtain

$$g(\zeta) - g(\eta) = (\zeta - \eta)\alpha g(\zeta)\{g(\zeta) - (\zeta - \eta)\alpha g(\zeta)g(\eta)\}.$$

Then we divide this equation by $\zeta - \eta$ and take traces to obtain

$$\frac{f(\zeta) - f(\eta)}{\zeta - \eta} - \operatorname{tr} \alpha g(\zeta)^2 = -(\zeta - \eta) \operatorname{tr} \alpha^2 g(\zeta)^2 g(\eta).$$

Finally since $|\operatorname{tr} \gamma| \leq |\gamma|$ we conclude from the boundedness of $|g(\eta)|$ in a neighborhood of ζ that

$$\lim_{\eta \rightarrow \zeta} \frac{f(\zeta) - f(\eta)}{\zeta - \eta} = \operatorname{tr} \alpha g(\zeta)^2.$$

Hence $f(\zeta)$ is an entire function with $f'(\zeta)$ given by $f'(\zeta) = \operatorname{tr} \alpha g(\zeta)^2$.

Now we compute the Taylor series for f about the origin. Since $f(\zeta) = \operatorname{tr} (1 - \zeta \alpha)^{-1}$ what we clearly expect here is that for ζ small we can write $(1 - \zeta \alpha)^{-1}$ as the sum of an appropriate geometric series and then obtain f by taking traces. To be more precise and rigorous set

$$s_n(\zeta) = \sum_{i=0}^n \zeta^i \operatorname{tr} \alpha^i.$$

Then

$$\begin{aligned} f(\zeta) - s_n(\zeta) &= \operatorname{tr} \left\{ g(\zeta) - \sum_{i=0}^n \zeta^i \alpha^i \right\} \\ &= \operatorname{tr} g(\zeta) \left\{ 1 - (1 - \zeta \alpha) \sum_{i=1}^n \zeta^i \alpha^i \right\} \\ &= \operatorname{tr} g(\zeta) \zeta^{n+1} \alpha^{n+1} \end{aligned}$$

and thus

$$|f(\zeta) - s_n(\zeta)| \leq |\zeta|^{n+1} |\alpha|^{n+1} |g(\zeta)|.$$

Now by our previous remarks $|g(\zeta)|$ is bounded in a neighborhood of zero and hence for ζ sufficiently small we deduce that

$$\lim_{n \rightarrow \infty} s_n(\zeta) = f(\zeta).$$

We have therefore shown that

$$f(\zeta) = \sum_{i=0}^{\infty} \zeta^i \operatorname{tr} \alpha^i$$

is the Taylor series expansion for $f(\zeta)$ in a neighborhood of the origin. Furthermore, f is an entire function and hence we can invoke a well-known theorem from complex analysis ([1, Theorem 3, pg. 142]) to deduce that the above series describes $f(\zeta)$ and converges for all ζ . In particular we have

$$\lim_{n \rightarrow \infty} \operatorname{tr} \alpha^n = 0$$

and this holds for all $\alpha \in JC[G]$.

We conclude the proof by showing that if $JC[G] \neq 0$ then there exists an element $\alpha \in JC[G]$ which does not satisfy the above. Indeed, suppose β is a nonzero element of $JC[G]$ and set $\alpha = \beta \beta^* / \|\beta\|^2$. Then $\alpha \in JC[G]$ since the Jacobson radical is an ideal. Furthermore we have $\alpha = \alpha^*$ and

$$\operatorname{tr} \alpha = \|\beta\|^{-2} \operatorname{tr} \beta \beta^* = \|\beta\|^{-2} \|\beta\|^2 = 1.$$

Now the powers of α are also symmetric under $*$ so for all $m \geq 0$ we have

$$\operatorname{tr} \alpha^{2m+1} = \operatorname{tr} \alpha^{2m} (\alpha^{2m})^* = \|\alpha^{2m}\|^2 \geq (\operatorname{tr} \alpha^{2m})^2.$$

Hence by induction $\text{tr } \alpha^{2^m} \geq 1$ for all $m \geq 0$ and this contradicts the fact that $\text{tr } \alpha^n \rightarrow 0$. Thus $JG = 0$ and the result follows.

Needless-to-say this analytic proof both excited and annoyed the algebraists who rightly viewed group rings as an algebraic subject. Later algebraic proofs were indeed given and the best result to date in characteristic 0 is as follows. Let K be a field of characteristic 0 which is not algebraic over the rationals Q . Then for all groups G , $K[G]$ is semisimple. Furthermore, for all the remaining fields the semisimplicity question is equivalent to that of the rational group ring (see [5], Theorem 18.3).

We should mention at this point that there is an amusing ploy to try to extend the above argument to $Q[G]$. Namely, we again consider $f(\zeta) = \text{tr}(1 - \zeta\alpha)^{-1}$, this time as a map from Q to Q , and we observe as before that for $|\zeta|$ small

$$f(\zeta) = \sum_0^\infty \zeta^i \text{tr } \alpha^i.$$

Now the right hand side here describes an analytic function in a neighborhood of the origin and this function has the rather strange property that it takes rationals to rationals. The question then is: must such a function necessarily be a polynomial? Unfortunately this is not the case and a simple counterexample is as follows. Let $r_1, r_2, \dots, r_n, \dots$ be an enumeration of the nonnegative rationals and define

$$h(\zeta) = \sum_{n=1}^\infty \frac{1}{n!} \frac{(r_1^2 - \zeta^2)(r_2^2 - \zeta^2) \cdots (r_n^2 - \zeta^2)}{(r_1^2 + 1)(r_2^2 + 1) \cdots (r_n^2 + 1)}.$$

Then h is easily seen to be an entire function which takes Q to Q and which is not a polynomial.

Actually, it now seems that the semisimplicity problem for $Q[G]$ will be settled by purely ring theoretic means since the result will follow from an appropriate noncommutative analog of the Hilbert *Nullstellensatz*. To be more precise, it has been conjectured that the Jacobson radical of a finitely generated algebra is always a nil ideal. This conjecture, along with the known fact ([5] Theorem 18.5) that $Q[G]$ has no nonzero nil ideals, would then easily yield $JQ[G] = 0$. Thus what is really of interest is the case of fields of characteristic $p > 0$ and here we are just beginning to guess at what the answer might be. Suppose first that G is finite. Then $K[G]$ is semisimple here if and only if $p \nmid |G|$. Of course this latter condition does not make sense for infinite groups, but the equivalent condition, namely that G has no elements of order p , does. None-the-less this is not the right answer. It is not true that $JK[G] \neq 0$ if and only if G has an element of order p . What does seem to be true is that elements of order p do play a role, provided that they are suitably well placed in G . To give this ambiguous idea some more meaning we start by quoting a simple fact ([5] Lemmas 16.9 and 17.6).

LEMMA 6. *Let α be an element of the group ring $K[G]$. Then $\alpha \in JK[G]$ if and only if $\alpha \in JK[H]$ for all finitely generated subgroups H of G with $H \supseteq \text{Supp } \alpha$.*

This of course says that whatever "well placed" means exactly, it must surely depend upon how the particular element sits in each finitely generated subgroup of G . Now we get more specific. Let H be a normal subgroup of G . Then we say that H carries the radical of G if

$$JK[G] = JK[H] \cdot K[G].$$

Our goal here is to find an appropriate carrier subgroup H of G such that the structure of $JK[H]$ is reasonably well understood. Admittedly this is a somewhat vague statement but we would certainly insist that $JK[H]$ be so simple in nature that we can at least decide easily whether or not it is zero.

The first candidate for H is based upon the Δ^+ subgroup and Lemma 6 above. We define

$$\Lambda^+(G) = \{x \in G \mid x \in \Delta^+(L) \text{ for all finitely generated subgroups } L \text{ of } G \text{ containing } x\}.$$

Then $\Lambda^+(G)$ is a characteristic subgroup of G and in all of the examples computed so far $\Lambda^+(G)$ does

indeed carry the radical. Furthermore, the ideal

$$I = JK[\Lambda^+(G)] \cdot K[G]$$

has been studied in general and it has been found to possess many of the properties expected of the Jacobson radical. Thus it now appears that $\Lambda^+(G)$ carries the radical but we are unfortunately a long way from a proof of this fact. Finally we remark that this conjecture has a nice ring theoretic interpretation. Namely it is equivalent to the assertion that if G is a finitely generated group, then $JK[G]$ is a union of nilpotent ideals.

Now given the above conjecture, the next natural step is to study $\Lambda^+(G)$ and $JK[\Lambda^+(G)]$ and it soon becomes apparent that these objects are not as nice as we had hoped for. Indeed $\Lambda^+(G)$ turns out to be just any locally finite group and so the problem of determining $JK[\Lambda^+(G)]$ is decidedly nontrivial. We are therefore faced with the problem of studying locally finite groups in general and here a new ingredient comes into play.

Let G be a locally finite group so that by definition all finitely generated subgroups of G are finite. Now if A is such a finite subgroup of G we say that A is locally subnormal in G if A is subnormal in all finite groups H with $A \subseteq H \subseteq G$. In other words, we demand that each such H has a chain of subgroups.

$$A = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H$$

for some n with H_i normal in H_{i+1} . Then using these locally subnormal subgroups as building blocks in a certain technical manner we can define a new and interesting characteristic subgroup of G denoted by $\mathcal{S}(G)$. Again it turns out that in all computed examples $\mathcal{S}(G)$ carries the radical of G and that furthermore the ideal

$$I = JK[\mathcal{S}(G)] \cdot K[G]$$

possesses in general many properties expected of the Jacobson radical. Thus it now appears that if G is locally finite then $\mathcal{S}(G)$ carries the radical of G .

The semisimplicity problem for groups in general has therefore been split into two pieces, namely the cases of finitely generated groups and of locally finite groups. Moreover, if we combine the corresponding conjectures for each of these cases, then what we expect, or at least hope, is that the group $\mathcal{S}(\Lambda^+(G))$ carries the radical of G . While this group may appear to be somewhat complicated in nature, there are in fact some nice structure theorems for it. Furthermore, and most important, we have $JK[\mathcal{S}(\Lambda^+(G))] \neq 0$ if and only if $\mathcal{S}(\Lambda^+(G))$ contains an element of order p .

Added in Proof. Hopefully reference [4] will appear soon in translation in the Journal of Soviet Mathematics. There is a new monograph in Russian by A. A. Bovdi entitled *Group Rings*. A second volume is anticipated. This author is presently working on an expanded and updated version of [5] which will be entitled *The Algebraic Structure of Group Rings*. It will be published in two volumes by Marcel Dekker, Inc. There has been some exciting progress recently on the zero divisor problem, in particular in the case of polycyclic-by-finite groups. This can be found in the papers by K. A. Brown, *On zero divisors in group rings*, to appear in the Journal of the London Math Society, and by D. R. Farkas and R. L. Snider, *K_0 and Noetherian group rings*, to appear in the Journal of Algebra.

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MATHEMATICAL NOTES

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A CONNECTED, LOCALLY CONNECTED, COUNTABLE HAUSDORFF SPACE

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In this MONTHLY, 76 (1969) 169–171, A. M. Kirch gave an example of a connected, locally connected countable Hausdorff space. We give another, geometrically much simpler, example. The example we provide will be a modified version of a connected, but not locally connected, countable Hausdorff space constructed by R. H. Bing in [1].

Let $X = \{(a, b) \in Q \times Q \mid b > 0\}$, where Q denotes the set of rational numbers, and let θ be a fixed irrational number. The points of our space will be the elements of X . With each $(a, b) \in X$ and each $\varepsilon > 0$ we associate two sets $L_\varepsilon(a, b)$ and $R_\varepsilon(a, b)$ which consist of all points of X in the triangles ABC and $A'B'C'$, respectively, where $A = (a - b/\theta, 0)$, $B = (a - b/\theta + \varepsilon, 0)$, $C = (a - b/\theta + \varepsilon/2, \theta\varepsilon/2)$, $A' = (a + b/\theta - \varepsilon, 0)$, $B' = (a + b/\theta, 0)$, and $C' = (a + b/\theta - \varepsilon/2, \theta\varepsilon/2)$. We now define a basic ε -neighborhood of $(a, b) \in X$ by $N_\varepsilon(a, b) = \{(a, b)\} \cup L_\varepsilon(a, b) \cup R_\varepsilon(a, b)$.

Since θ is irrational, no two points of X can lie on a line with slope θ or $-\theta$. Thus, for any two distinct points of X , we may choose a sufficiently small $\varepsilon > 0$ such that the points lie in disjoint ε -neighborhoods. Therefore, X is Hausdorff and, obviously, a countable space.

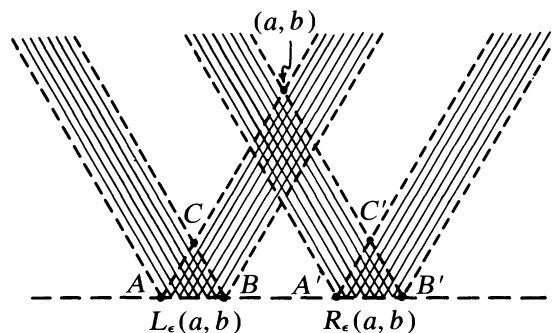


FIG. 1

Figure 1 shows that the closure of each basic ε -neighborhood $N_\varepsilon(a, b)$ consists of the union of four infinite strips with slope $\pm \theta$ emanating from the bases of the triangles $R_\varepsilon(a, b)$ and $L_\varepsilon(a, b)$. Hence the closures of each pair of basic neighborhoods intersect. Therefore, X is connected but not regular and, hence, not metrizable.