Seeking Relevance? Try the History of Mathematics

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What is mathematics? This seems like a curious question to pose to a read ing audience of mathematics educators, but I would wager that, on reflection, it is a difficult one to answer. Of course, a variety of quips and clichés can supply the answer: "It's an art," "A science—in fact, it's the queen and the servant of science," "It's what I use when I balance my checkbook," "A game that we play with rules we're not quite sure of," and the apologetic favorite, "Something I was never good at"; and so the list can go on and on. What these answers seem to avoid is the fact that mathematics is a necessary human activity, one that reflects a response to needs dictated by human existence itself. Mathematics is evident in all societies and cultures. The needs to which it responds are both material and intellectual, and as they change, so does the nature of the mathematics that serves them; thus, mathematics is a body of knowledge that is constantly evolving in response to societal conditions.

Just as each of the responses given in reply to the initial question is one dimensional, so, all too often, is our teaching of mathematics one dimensional. We frequently find ourselves concentrating on the teaching of "mathematics"—the symbols, the mechanics, the answer-resulting procedures—without really teaching what mathematics is "all about"—where it comes from, how it was labored on, how ideas were perceived, refined, and developed into useful theories—in brief, its social and human relevance. At best, this practice will produce knowledgeable technicians who can dispassionately use mathematics, but it will also produce students who perceive mathematics as an incomprehensible collection of rules and formulas that appear en masse and threateningly descend on them.

Unfortunately, in our schools, this latter group is in the majority. These students build psychological barriers to true mathematical understanding and develop anxieties about the learning and use of mathematics. Teachers can partially remedy this situation by incorporating a historical perspective into the teaching of mathematics. History is commonly taught in schools to initiate the young into a community—to give them an awareness of tradition, a feeling of belonging, and a sense of participation in an ongoing process or institution. Similar goals can be advocated for the teaching of the history of mathematics. By incorporating some history into teaching mathematics, teachers can lessen its stultifying mystique. Mathematics isn't something magic and forbiddingly alien, but rather it's a body of knowledge naturally developed by people over a 5000-year period—people who made mistakes and were often puzzled but who worked out solutions to their problems and left records of those solutions so that we can benefit from them. Its teaching should recognize and promote these people-centered facts.

Certainly, the chronicles of mathematics possess all the facets of human drama that capture the imagination and perpetuate interest: mystery, adventure, intrigue, and so on. High drama, as well as a nice introduction to the theory of mathematical modeling, can be found in a consideration of Galileo's challenge to the Ptolemic theory of planetary motion. Mystery abounds in numerology, the construction and purpose of the Egyptian pyramids, and the appearance of mathematical constants such as pi and e in diverse phenomena. The historical search for higher order constructible regular polygons provides adventure. Intrigue can be found in "Dungeon and Dragon"-type secret numerical codes that were popular in the Middle Ages or in the appearance of descriptive geometry in 1789 as a military weapon. The history of mathematics can convey the heights of human ingenuity and creative genius—for example, Cantor's diagonal proof of the uncountability of the real numbers—or demonstrate the foibles of human understanding as illustrated by the continued existence of proofs of the trisectibility of an angle. Such material can breathe life into mathematics lessons.

Incorporating History into Mathematics Teaching

Mathematics teaching can thus be humanized by the inclusion of historical perspectives in classroom discussions. This can, and should, be done unobtrusively by the use of historical anecdotes, films, projects, displays, and problems in teaching presentations as well as the replication of relevant historical experiments, e.g., the estimation of π by a method of exhaustion (NCTM 1969). It should be done in a natural way as an integral part of the lesson—no announcements to the effect that "now we're going to talk about the history of mathematics" should take place!

Consider, for example, how lessons on the development of a numeration system might be historically enhanced. A discussion can point out how our present system of numerals evolved from simple tally symbols such as 1 = ≡ and how writing familiarity and a quest for speed transformed these symbols to 1 ≥ ≷ . The evolution of arabic numerals from such primitive tally strokes to the modern liquid-crystal displays of hand-held calculators or the numerals on bank checks can be demonstrated by the use of a simple chart (fig. 1). Are the black lines employed by optical scanners to read the price of a box of soap powder the modern equivalent of tally strokes?

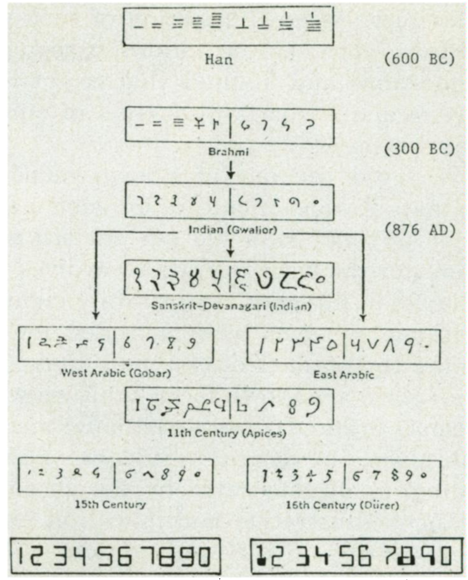


Fig. 1. An evolution of numerals

Frequently in the teaching of number bases, references are made to Egyptian and Babylonian number systems. Instead of just talking about hieroglyphic and cuneiform numerals, one can generate much more excitement by giving students an opportunity actually to translate them from facsimilies of ancient records. A minimum student knowledge of hieroglyphic numerals and guided inspection of the contents of figure 2 could reveal the fact that the ancient Egyptians knew the formula for the computation if the volume of the frustum of a square pyramid, i.e.,

$$V = \frac{1}{3}h(a^2 + ab + b^2),$$

where a and b are measures of the sides of the bases and h a measure of the height.

As a result of such activities many questions arise. Each question, itself, offers new learning opportunities, and its answers can lead to profound insights into the growth process of mathematical ideas and

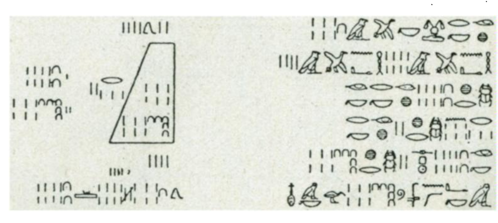


Fig. 2. Egyptians knew formulas for the volume of pyramids.

techniques. As an illustration of such possibilities, consider the following sequence of questions and findings that might logically emanate from a discussion of numeration systems.

Why did the use of Roman numerals dominate European culture for such a long period of time? How did our modern algorithms for the basic operations evolve? For example, in fifteenth-century Italy eight algorithmic schemes were accepted for obtaining the product of two multidigit numbers. (Some are shown in fig. 3.) Even when attempts to standardize a particular method took place, the desired result was slow in coming, as demonstrated by the format of the same illustrative multiplication example appearing in different editions of Taglient's *Libro Dabaco* over a fifty-year period (fig. 4).

(1515)	(1520)	(1541)	(1550)	(1567)
456	456	456	456	456
23	23	23	23	23
1368	1368	136 8	136 8	1368
912	912	912	912	912
10488	10488	1048 8	104 88	10488

Fig. 4. Attempts to standardize an algorithm

Was such confusion the result of the printer's incompetence, or does it reflect the author's continuing modification of the algorithmic form? What factors contributed to the popularization of an algorithm? Consider the case of the "galley," or "battelo," method of division that was favored among Renaissance mathematicians. Although it was mathematically efficient, it was not popular with printers who had the task of setting the type to reproduce it; thus, the algorithm fell into disuse (fig. 5).

Examinations of the content of the history of mathematics frequently reveal striking similarities to present-day mathematical situations and procedures. How were square roots handled in antiquity? See figure 6. It is interesting to note that the Babylonians used a "divide and average" method of finding square roots: Given a number x, we wish to find the square root of x. Guess at the root and call it a. Then use the following procedure:

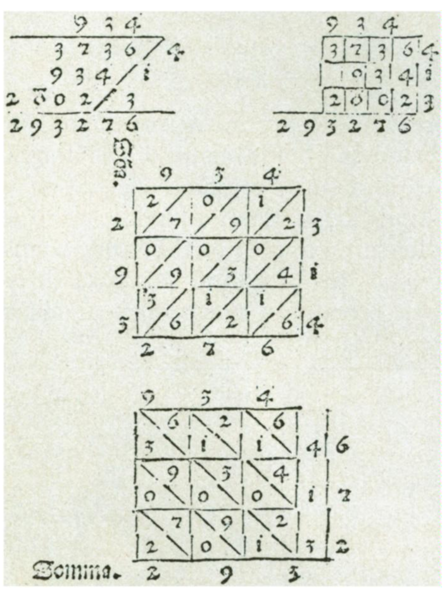


Fig. 3. Some multiplication algorithms

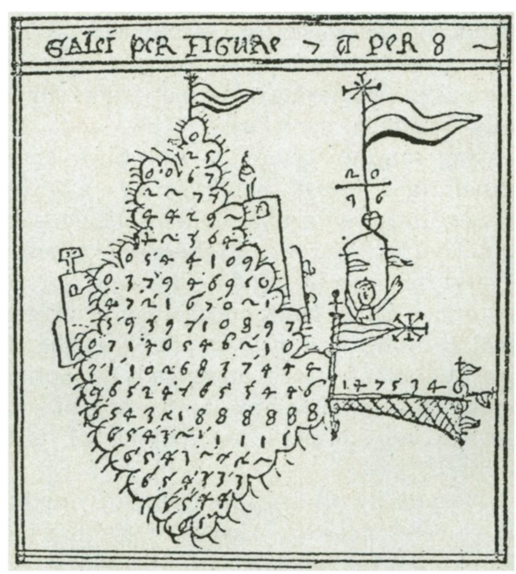


Fig. 5. "Galley" division

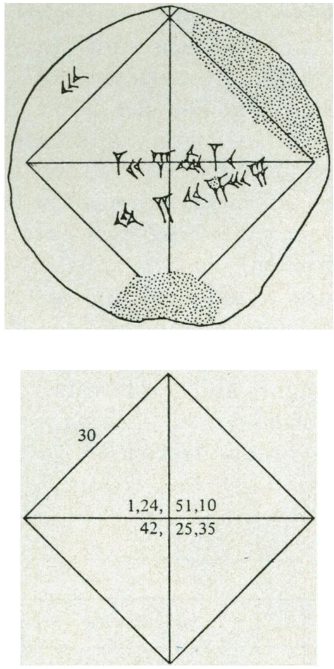


Fig. 6. Finding square roots

$$\frac{x}{a_1} = r_1, \frac{r_1 + a_1}{2} = a_2,$$
$$\frac{x}{a_2} = r_2, \frac{r_2 + a_2}{2} = a_3,$$
$$\frac{x}{a_3} = r_3, \frac{r_3 + a_3}{2} = a_4.$$

Continue this process until the desired degree of accuracy is achieved. What makes the technique even more interesting is that it is one of the earliest examples of an iterative process that we know, and iterative computing processes lie at the heart of modern computer operations. Our mathematical continuity with the past is ever present. In particular, iteration and approximation have always been and always will be integral components of computational processes. When discussing the history of mathematics, it is important to recognize the role of approximation in the efforts of human beings to quantify their world. "How good and useful were such approximations?" is a question that should frequently be brought before students. For example, Chinese mathematicians of the early Han dynasty (ca. 300 B.C.) used a trapezoidal approximation for the area A of the segment of a circle,

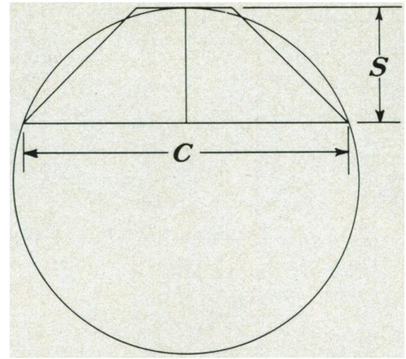
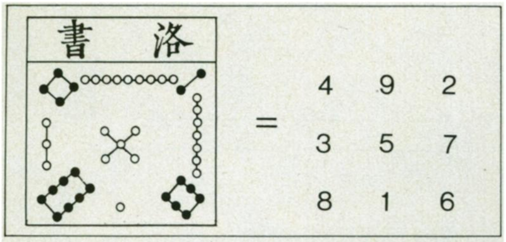


Fig. 7. A trapezoidal approximation for area

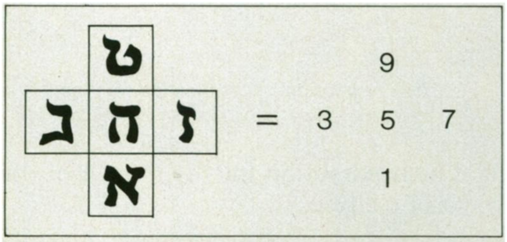
$$A = \frac{s}{2}(C + S),$$

where C is the measure of the length of the chord involved and S is the measure of the length of the sagitta of the segment (fig. 7). How good is this approximation?

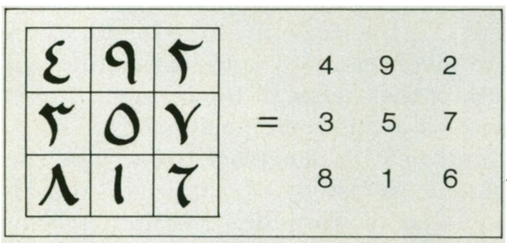
Mathematics has sometimes been described as a "study of patterns." History would seem to affirm this description. Numbers and number patterns, especially those that demonstrated an emergence of regularity out of apparent disarray, fascinated early observers, and magical number meanings and configurations were created. The magic square exemplifies this facet of mathematical history. See figure 8. The Chinese saw in their *Lo shu* magic square (8a) the origins of science and mathematics, whereas the Hebrews used a serrated version of the same square to present the sacred name "Yahweh" in coded form (8b); in turn this square found its way into Islam, where it became a talisman painted on dinner plates that were sold to Europeans to ward off the Black Death (8c). Although magic squares and number configurations began as symbolic devices of supernatural significance, they eventually evolved into intellectual challenges, intriguing such notable personages as Benjamin Franklin (fig. 9a), and were used as schemes to sharpen inductive reasoning powers and build problem-solving skills



(a) Lo shu



(b) Yahweh



(c) "Clean plate" talisman

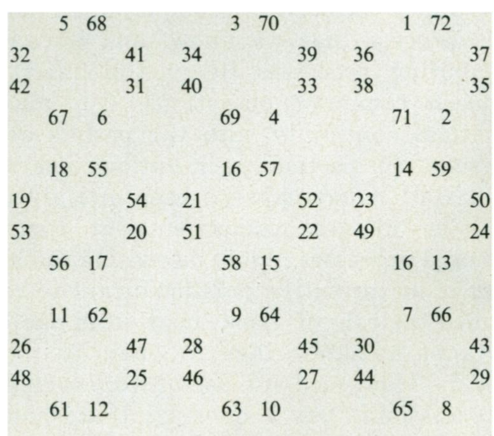
Fig. 8

among Chinese mathematicians of the Ming dynasty (fig. 9b). Their appeal and computational attraction are equally relevant to our students, who can explore and unravel their mysteries with the assistance of handheld calculators.

Frequently, historical material furnishes truly elegant examples and explanations that can be adopted for classroom use. Consider the traditional and universally popular "bride's chair" proof of the Pythagorean theorem shown in figure 10. Although this proof is mathematically appealing, its level of sophistication limits its pedagogical usefulness. At times it has been described as a "mouse trap proof" and "a proof walking on stilts, nay, a mean, underhanded proof" (Kline 1962, p. 50). An alternative proof, one that lends itself nicely to an overlay-overhead projector demonstration, is the "husan-thu" proof offered by Chinese mathematicians of the early Han period (ca. 300 B.C.). See figure 11. Throughout the history of mathematics, a variety of such dissection proofs have been offered to affirm the proposition that given a right triangle whose legs have measures of length a units and b units, respectively, and whose hypotenuse has a measure of length c units then $a^2 + b^2 = c^2$ (Loomis 1968). Over the years people with many backgrounds and interests shared a common obsession in seeking a solution to the Pythagorean theorem.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

(a) Franklin 8 × 8 square



(b) Magic circles from Ming period

Fig. 9

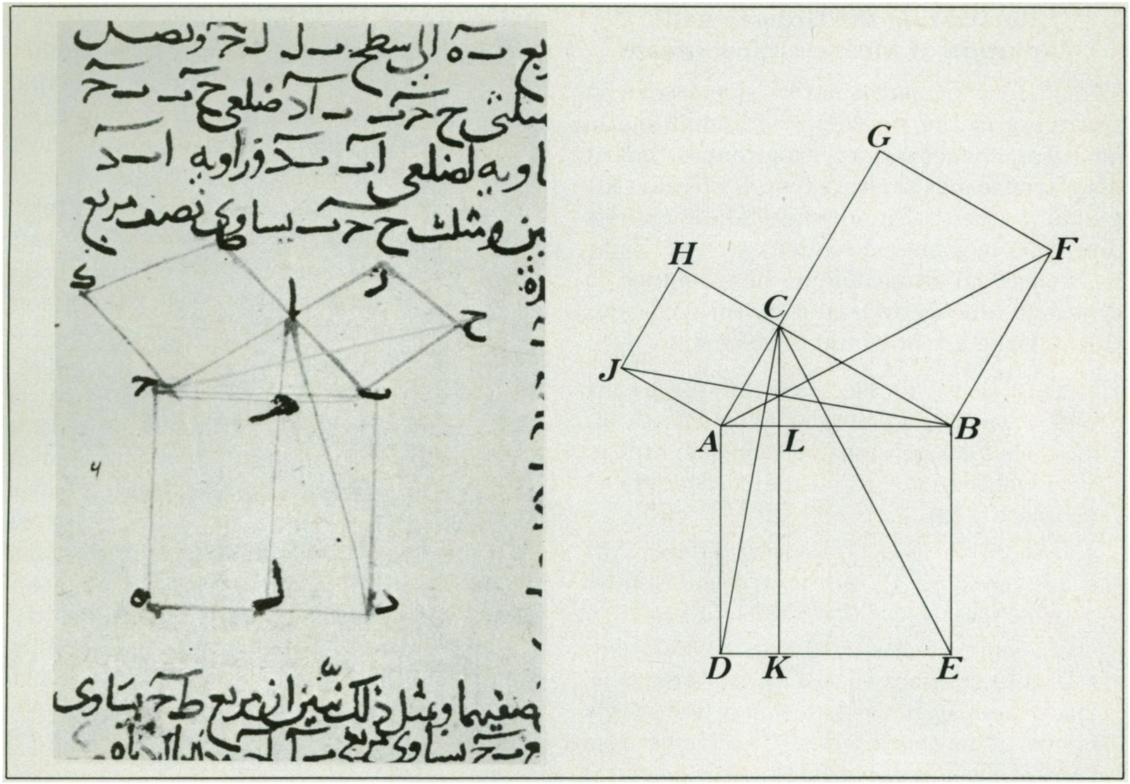
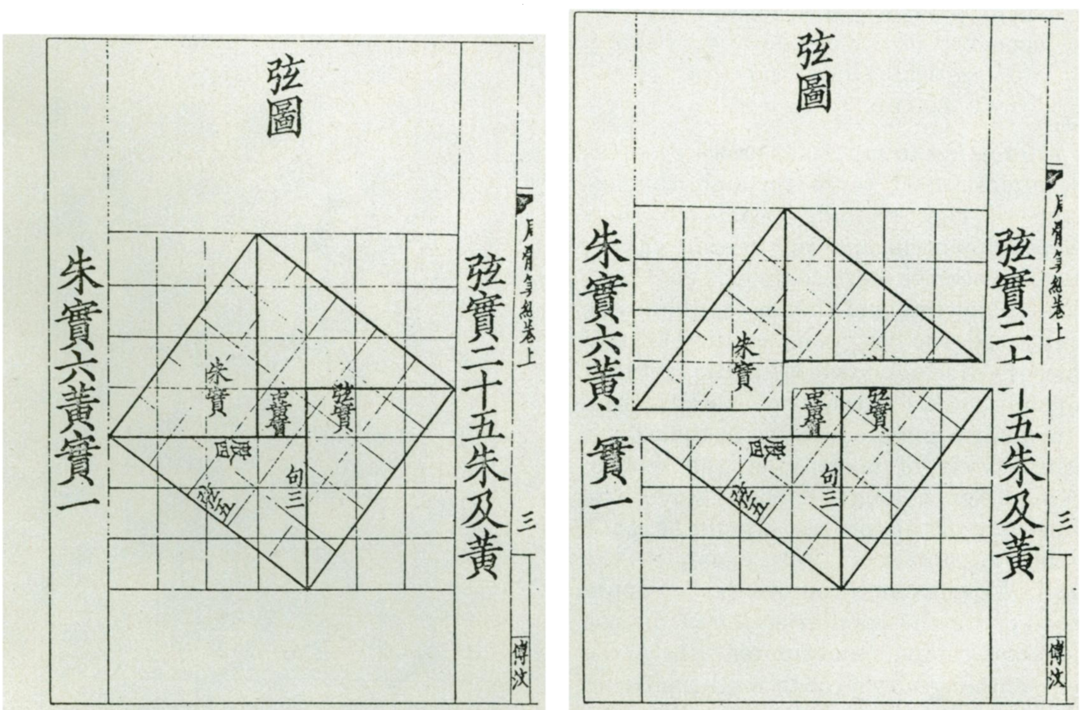


Fig. 10



(a) Square of area c^2 composed of four right triangles whose legs measure a and b units, hypotenuse c units, plus a small square in the center

(b) Pieces rearranged, thus $c^2 = a^2 + b^2$.

Fig. 11

Illustrating the Growth and Evolution of Mathematical Ideas

The history of mathematics spans time in testifying to the similarity of human mathematical concerns and experiences, but it also transcends cultures and affirms the global nature of mathematical awareness and involvement. Frequently, the same mathematical situation can be examined in different time periods and cultural milieus through the use of extant problems.

> When a ladder 25 feet long rests against the vertical wall of a building, its top is 17 feet farther from the base of the building than its foot. How high on the wall does the ladder reach? (Dolciani 1970)
>
> A spear 20 feet long leans against a tower. If its end is moved out 12 feet, how far up the tower does the spear reach? (Italy, 1300 A.D.)
>
> The height of a wall is 10 feet. A pole of unknown length leans against the wall so that its top is even with the top of the wall. If the bottom of the pole is moved 1 foot farther from the wall, the pole will fall to the ground. What is the height of the pole? (China, 300 B.C.)
>
> A beam of length 30 feet stands against a wall. The upper end has slipped down a distance 6 feet. How far did the lower end move? (Babylonia, 1600–1800 B.C.)

It is interesting to note that as we progress back almost 4000 years through this sequence of right-triangle problems, their conceptual content does not become easier; in fact, it becomes more complex.

When Newton said that he could devise calculus because he "stood on the backs of giants," he was acknowledging the fact that mathematical discoveries are usually gradual in coming and are assisted in their birth by the efforts of many men and women working over a long period of time. The gradualness of this process should be made evident to students. For example, the phrase "Cartesian coordinates" would seem to credit the popularization of the use of rectangular coordinates to the seventeenth-century French mathematician and philosopher René Descartes. Was Descartes the originator of such an idea? No, it does not seem so, for systems of rectangular coordinates were employed by Renaissance artists as a technical aid in achieving perspective in their paintings; by early Greek, Roman, and Chinese cartographers who used accurate rectangular grid systems in their map designs; and by Egyptian tomb painters who as early as the XVIII dynasty (1552–1306 B.C.) constructed graphical grids to assist in the transfer of sketches from their working tablets to tomb walls. See figure 12.

Time and time again in the history of mathematics, a similar scenario unfolds, a scenario that shows how mathematics evolved as a science: a mathematical idea or concept is pursued and studied for its utilitarian or sociological significance, but gradually its considerations become abstract and divorced from empirical reality. An example of this phenomenon is illustrated by the three classical problems of Greek antiquity:

1. The duplication of the cube
2. The trisection of an angle
3. The quadrature of the circle

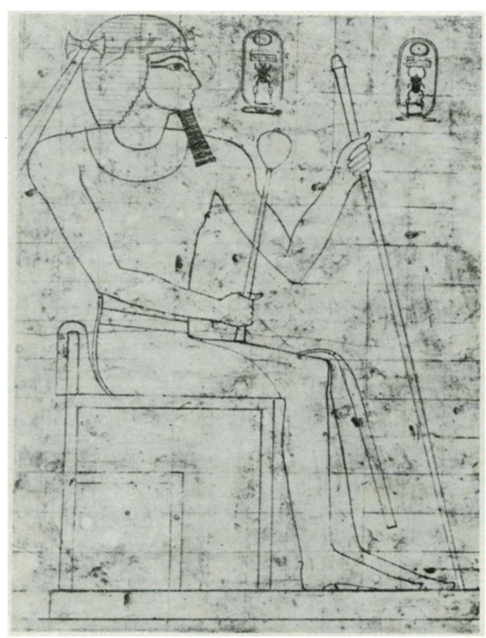
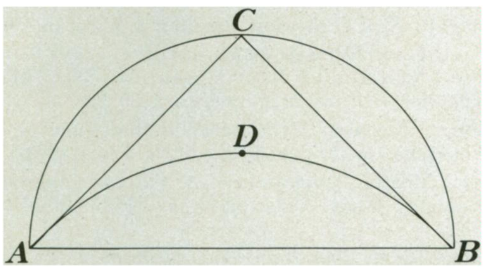


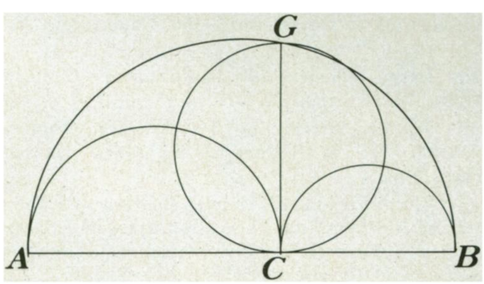
Fig. 12. Grids help transfer sketches to tomb walls.

These problems were to be solved with the use of a straightedge and compass alone. It was over two thousand years before this feat was proved impossible, but yet, in the interim search for solutions, many useful mathematical discoveries were made, including a theory of conic sections and the development of cubic, quadratic, and transcendental curves. Out of this particular legacy emerged a series of geometric problems that can challenge and fascinate modern students. In the search to achieve a quadrature of the circle, a theory of lunes developed and problems like those that follow resulted.

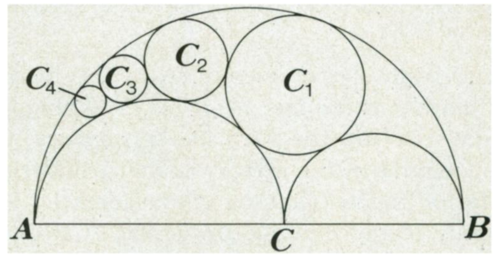


Given a semicircle with diameter AB, arc ADB is inscribed in the semicircle. The region bounded between the semicircle and the arc is called a lune. Show that the area of the lune $ACBD$ is equal to the area of the inscribed triangle ACB where $\overline{AC} \cong \overline{CB}$.

Problems become more complex and further removed from reality, as shown by a consideration of the arbelos and its properties (Raphael 1973):



Let A, C, and B be three points on a straight line. Construct semicircles on the same side of the line with AB, AC, and CB as diameters. The region bounded by these three semicircles is called an *arbelos*. At C construct a perpendicular line to AB intersecting the largest semicircle at point G. Show that the area of the circle constructed with CG as a diameter equals the area of the arbelos.



Given an arbelos packed with circles C_1, C_2, C_3, ..., as indicated, show that the perpendicular distance from the center of the nth circle to the line ACB is n times the diameter of the nth circle.

In tracing the cultural, temporal, and intellectual migration of mathematical concepts and techniques, it is important that the social, political, and economic forces that have influenced the growth of mathematics be recognized and acknowledged. Just why is it that the rise of deductive mathematics appeared in classical Greece? Why did the emergence of mercantile capitalism in fourteenth-century Italy spawn a mathematical Renaissance? How did the invention of printing affect the growth and spread of mathematical ideas? Is a nation's mathematical competency contingent on its wealth? Does war change the nature of a country's mathematical activity? It is from the contemplation of such questions and their answers that the dynamic interrelationship of mathematics and society emerges.

Conclusion

A recognition of the pedagogical importance of the history of science and mathematics in teaching is certainly not new. George Sarton, the noted historian of science, frequently pleaded this cause. In addressing a general audience in 1953, Sarton delivered a message that is quite appropriate today:

I said that if you do not love and know science, one cannot expect you to be interested in its history; on the other hand, the teaching of the humanities of science would create the love of science as well as a deeper understanding of it. Too many of our scientists (even the most distinguished ones) are technicians and nothing more. Our aim is to humanize science, and the best way of doing that is to tell and discuss the history of science. If we succeed, men of science will cease to be mere technicians, and will become educated men (Sarton 1958).

Unfortunately, in recent years the history of mathematics has been experiencing neglect. Around the turn of the century, the importance of the history of mathematics in relation to the teaching of mathematics was widely recognized and promoted. Almost all universities and teacher-training colleges offered studies in the subject, but gradually its status diminished as mathematical content, itself, became the sole focus of mathematics teaching. A survey of the offerings of American colleges and universities in 1980–81 revealed that only 30 percent offered a course on the history of mathematics (ISGHPM 1982). In light of this revelation, it is rather ironic to note that during this same period many professional schools (law, medicine, architecture, and so on) are requiring their graduates to take courses in the history of their discipline. This rather recent curriculum innovation is intended to refocus attention and concern on the people-centered origins and applications of the disciplines in question. Perhaps it is also time to attempt such a reorientation in our teaching of mathematics.

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For additional suggestions see the bibliography "History and Cultural Evolution of Mathematics" in the *High School Mathematics Library* by William L. Schaaf (available from NCTM, stock #197, $7.80 for nonmembers, $6.24 for individual members).

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(Continued on page 47)

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An Astounding Revelation on the History of π

(Continued from page 52)

The Gaon of Vilna then found the ratio of these two values:

$$\frac{111}{106} = 1.0472$$

(to four decimal places), which he considered the necessary correction factor. When he multiplied this value by 3 (the previously thought-to-be value of π in the Bible), he obtained $3 \times 1.0472 = 3.1416$, the value of π correct to four decimal places!

This interpretation should then correct the notion that the Bible had only a very crude approximation of π. Mathematics historians, take note!

Seeking Relevance? Try the History of Mathematics

(Continued from page 62)

Swetz, Frank J., and T. I. Kao. *Was Pythagoras Chinese? An Examination of Right Triangle Theory in Ancient China.* University Park, Pa.: Pennsylvania State University Press; Reston, Va.: National Council of Teachers of Mathematics, 1977.

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Answers for "Problems of the Month" on page 37

1. Let f be the number of strides made by the father and s be the number of strides by the son after the head start. From the information given in the problem,

$$\frac{s}{9} = \frac{f}{6},$$

or

$$s = \frac{3}{2}f. \tag{1}$$

Let D be the distance in each of dad's strides and d be the distance in each of the son's strides. From the information given in the problem,

$$3D = 7d,$$

or

$$D = \frac{7}{3}d. \tag{2}$$

Now, $f \cdot D$ is the distance father goes, $s \cdot d$ is the distance son goes after father starts running, and $60 \cdot d$ is the distance in son's head start. From the problem,

$$f \cdot D - s \cdot d = 60 \cdot d. \tag{3}$$

Substitution of (1) and (2) into (3) gives an equation reducible to one variable, from whence $f = 72$ and $s = 108$. The father makes 72 strides, and the son makes 108 strides.

2. We can set up a system of equations based on the fact that

production cost × rate profit = selling price.

In this problem, production cost must be refined to a product of cost per unit × number units (the unit being cubic inches in this problem). Thus, our equation before the hollowing out is as follows:

cost per unit × units × rate profit = selling price;

$$C \times \frac{\pi}{6} \times P = 1¢$$

In this equation, $\pi/6$ equals the volume of a solid gumball of diameter one inch.

Letting r be the radius of a concentric sphere that is removed from the gumball, producing the shell, we obtain our second equation:

$$8C \times \left[\frac{\pi}{6} - \frac{4\pi r^3}{3}\right] \times P = 2¢$$

(Remember, an increase of 700 percent is an *eightfold* increase!) Seeing from the first equation that the product $P \cdot C = 6/\pi$, we can substitute this value into the second equation to obtain the following:

$$\frac{48}{\pi}\left[\frac{\pi}{6} - \frac{4\pi r^3}{3}\right] = 2¢$$

From this equation, it follows algebraically that $r = \sqrt[3]{6}/4$. The thickness of the shell, therefore, is the original radius of one-half inch, less this value, which is

$$\frac{2 - \sqrt[3]{6}}{4}.$$

This result rounds off to a value of approximately 0.0457 inches thick, which would give a sensation similar to biting into a balloon! I was more concerned with finding numbers that worked out reasonably well than with realism....

3. -17 or 3