NOTES

Fitting One Right Triangle in Another

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In a survey article in this MAGAZINE [**2**], Wetzel posed several open questions about fitting one plane figure in another. When does a triangle fit in an equilateral triangle? When does a triangle fit in a rectangle? When does one right triangle fit in another right triangle? In this note, we answer the last question. This result constitutes part of the output of an undergraduate summer research project by the second author under the supervision of the first author.

We find conditions on a, b, c, d so that a right triangle with legs a, b (the initial triangle) fits in a right triangle with legs c, d (the target triangle). We may assume $a \leq b$ and $c \leq d$. First note some necessary conditions for a fit:

- (diameter condition) $\sqrt{a^2+b^2} \leq \sqrt{c^2+d^2}$;
- (thickness condition) $ab/\sqrt{a^2+b^2} \leq cd/\sqrt{c^2+d^2}$;
- (area condition) $ab \leq cd$.

(The third condition is a consequence of the first and second together.) Easy examples show that these conditions are not sufficient. To find necessary and sufficient conditions, we identify four cases:

(i) $a \leq c$ and $b \leq d$;
(ii) $a \geq c$ and $b \geq d$, but not both equalities hold;
(iii) $a < c$ and $b > d$;
(iv) $a > c$ and $b < d$.

Case (i). Here the initial triangle fits in the target triangle by aligning their right angles.

Case (ii). Here the area of the initial triangle exceeds that of the target triangle and no fit exists.

The two interesting cases remain. We may restrict our attention to the largest triangle of a given shape that fits in the target (for if a triangle fits in the target, any smaller triangle of the same shape will fit). The following result of Sullivan [**1**] simplifies the remaining cases:

Among the polygons inside a triangle T which are similar to a given polygon P, any largest one has two of its vertices along the same edge of T (and some vertex on each edge of T).

Case (iii): $a < c \leq d < b$. First note that for a fit to exist, we must have $b < \sqrt{c^2 + d^2}$. Next note that because b is longer than both c and d, we cannot get a fit by placing an edge of the initial triangle along a leg of the target triangle. Hence to get a fit, we place an edge of the initial triangle along the hypotenuse of the target. Now observe that an initial triangle with a leg placed along the hypotenuse can be reflected so that its hypotenuse lies along the hypotenuse and the reflected triangle remains inside the target. See FIGURE 1.

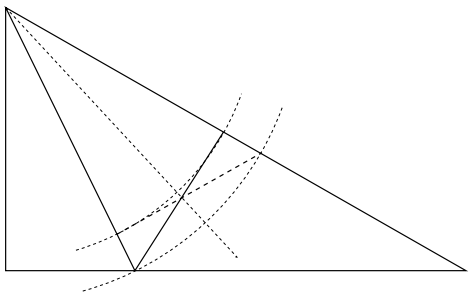

Figure 1

Hence we may assume that the hypotenuses of the triangles are aligned. This can be done in two ways as shown in FIGURE 2.

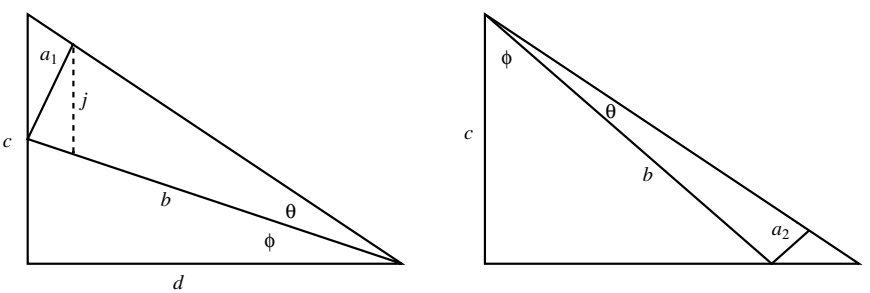

Figure 2

With b, c, d as given, we find the values of a_1 and a_2. Then a fit exists for any initial triangle whose leg a is at most the longer of a_1 and a_2.

In the triangle on the left, from the Law of Tangents,

$$\frac{a_1}{b} = \tan\theta = \tan\big((\theta+\phi)-\phi\big) = \frac{\tan(\theta+\phi)-\tan\phi}{1+\tan(\theta+\phi)\tan\phi}$$

$$= \frac{\frac{c}{d} - \frac{\sqrt{b^2-d^2}}{d}}{1+\frac{c}{d}\cdot\frac{\sqrt{b^2-d^2}}{d}} = \frac{d(c-\sqrt{b^2-d^2})}{d^2+c\sqrt{b^2-d^2}}.$$

Therefore,

$$a_1 = \frac{bd(c - \sqrt{b^2 - d^2})}{d^2 + c\sqrt{b^2 - d^2}}.$$

A similar computation gives

$$a_2 = \frac{bc(d - \sqrt{b^2 - c^2})}{c^2 + d\sqrt{b^2 - c^2}}.$$

(Or note that the difference between the two triangles is that the roles of c and d are interchanged.)

To compare a_1 and a_2, we compare numerators and denominators separately. To see that the numerator of a_2 is at most the numerator of a_1, observe that $d \geq c$ and $c^2 + d^2 > b^2$ imply $0 \leq (d^2 - c^2)(d^2 + c^2 - b^2) = d^4 - b^2d^2 + b^2c^2 - c^4$. Thus $\left(d\sqrt{b^2 - d^2}\right)^2 \leq \left(c\sqrt{b^2 - c^2}\right)^2$ implies

$$bc(d - \sqrt{b^2 - c^2}) \leq bd(c - \sqrt{b^2 - d^2}).$$

Next compare denominators:

$$\begin{aligned} 0 &\leq \left(d\sqrt{b^2 - d^2} - c\sqrt{b^2 - c^2}\right)^2 \\ &= d^2(b^2 - d^2) - 2cd\sqrt{b^2 - c^2}\sqrt{b^2 - d^2} + c^2(b^2 - c^2). \end{aligned}$$

Add $(d^2 - c^2)^2$ to get

$$\left(d^2 - c^2\right)^2 \leq \left(d\sqrt{b^2 - c^2} - c\sqrt{b^2 - d^2}\right)^2,$$

so

$$d^2 + c\sqrt{b^2 - d^2} \leq c^2 + d\sqrt{b^2 - c^2}.$$

Hence $a_2 \leq a_1$. Thus a fit exists if

$$b < \sqrt{c^2 + d^2} \quad \text{and} \quad a \leq \frac{bd(c - \sqrt{b^2 - d^2})}{d^2 + c\sqrt{b^2 - d^2}}.$$

The set of conditions that allow a fit in this case can be simplified by noting that the second inequality implies the first inequality (a must be positive). Also, the condition $a < c$ is not used in the derivation and is, in fact, a consequence. From the triangle on the left in FIGURE 2, we see $a \leq a_1 < j < c$.

Case (iv): $c < a \leq b < d$. Again we may assume the initial triangle fits in the target triangle so that their hypotenuses are aligned. And again there are two ways to do this as shown in FIGURE 3.

With a, b, c as given, we find d_1 and d_2. Then any right triangle whose leg d is at least the shorter of d_1 and d_2 will allow a fit. In the left triangle, using the Law of Tangents as above, we get

$$\frac{a}{b} = \frac{c(d_1 - \sqrt{b^2 - c^2})}{c^2 + d_1\sqrt{b^2 - c^2}}.$$

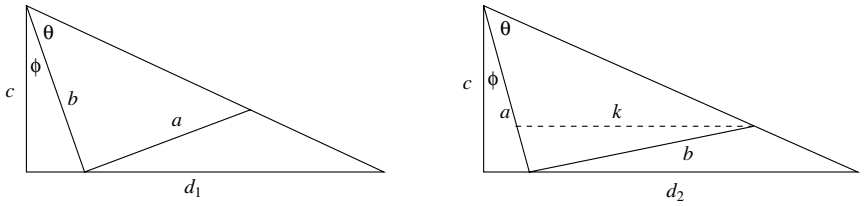

Figure 3

Solve for d_1 to get

$$d_1 = \frac{c(ac + b\sqrt{b^2 - c^2})}{bc - a\sqrt{b^2 - c^2}}.$$

Similarly, by interchanging the roles of a and b,

$$d_2 = \frac{c(bc + a\sqrt{a^2 - c^2})}{ac - b\sqrt{a^2 - c^2}}.$$

To compare d_1 and d_2, we rationalize the denominators:

$$d_1 = \frac{c(ab^3 + c(a^2 + b^2)\sqrt{b^2 - c^2})}{a^2c^2 + b^2c^2 - a^2b^2}, \quad d_2 = \frac{c(a^3b + c(a^2 + b^2)\sqrt{a^2 - c^2})}{a^2c^2 + b^2c^2 - a^2b^2}.$$

Now we need only compare numerators. Observe that $b \geq a$ implies

$$0 \leq ab(b^2 - a^2) + c(a^2 + b^2)\left(\sqrt{b^2 - c^2} - \sqrt{a^2 - c^2}\right).$$

Thus

$$c(a^3b + c(a^2 + b^2)\sqrt{a^2 - c^2}) \leq c(ab^3 + c(a^2 + b^2)\sqrt{b^2 - c^2}).$$

Hence $d_2 \leq d_1$. Thus a fit exists if

$$d \geq \frac{c(bc + a\sqrt{a^2 - c^2})}{ac - b\sqrt{a^2 - c^2}}.$$

If we solve this inequality for b, we get

$$b \leq \frac{ac(d - \sqrt{a^2 - c^2})}{c^2 + d\sqrt{a^2 - c^2}}.$$

As before, the inequality $b < d$, not used in the derivation, is a consequence (from the triangle on the right in FIGURE 3, we have $b < k < d_2 \leq d$).

Putting everything together, we can state our final result (where we choose to reverse the order of the last two cases):

THEOREM. *A right triangle with legs a, b ($a \leq b$) fits in a right triangle with legs c, d ($c \leq d$) if and only if*

(1) $a \leq c$ *and* $b \leq d$, *or*

(2) $a > c$ *and* $b \leq \dfrac{ac(d - \sqrt{a^2 - c^2})}{c^2 + d\sqrt{a^2 - c^2}}$, *or*

(3) $b > d$ *and* $a \leq \dfrac{bd(c - \sqrt{b^2 - d^2})}{d^2 + c\sqrt{b^2 - d^2}}$.

As an example of (2), we can take $a = 5$, $b = 6$, $c = 4$, $d = 78$; for (3), take $a = 18$, $b = 135$, $c = 106$, $d = 108$. In each case, the second inequality becomes an equality. See FIGURE 4.

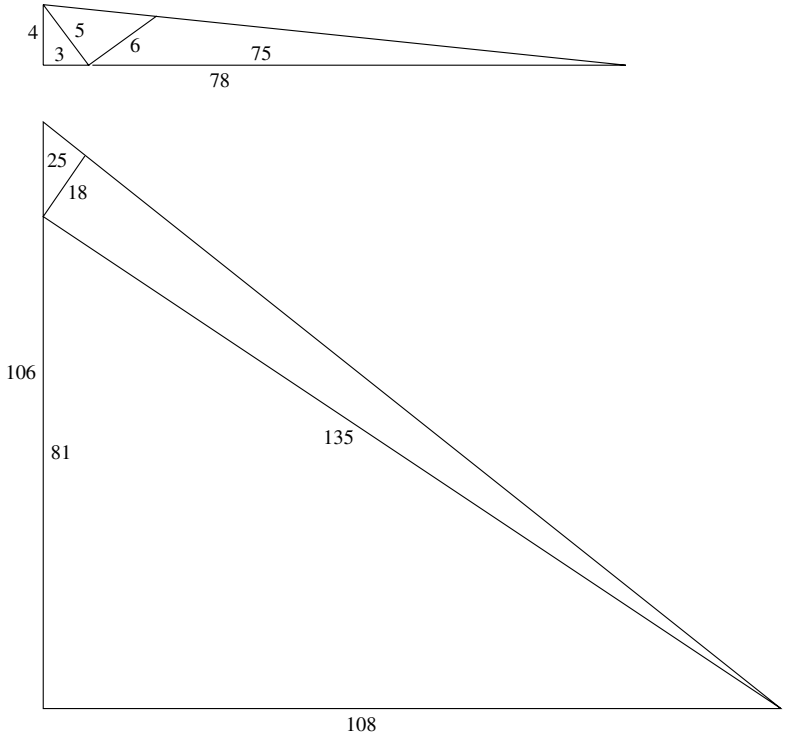

Figure 4

REFERENCES

1. J. M. Sullivan, Polygon in Triangle: Generalizing a Theorem of Post. (Preprint. http://torus.math.uiuc.edu/jms/Papers/post.pdf.)
2. J. E. Wetzel, Fits and Covers, this MAGAZINE **76** (2003) 349–363.

Determinants of Matrices over the Integers Modulo *m*

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As is proven in any elementary course on linear algebra, a square matrix over the real numbers is invertible if and only if its determinant is nonzero. For students who are familiar with $\mathbb{Z}_m$, the ring of integers modulo m, one can easily prove that a square