# CLASSROOM CAPSULES

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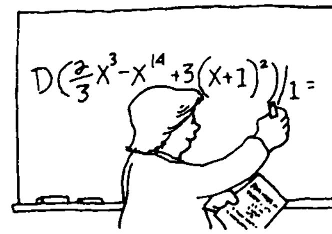


Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

## A Triple Angle Formula for Tangent

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As is well known,

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}. \tag{1}$$

In this capsule, we derive an alternative formula for $\tan 3\theta$ and show its applications.

**Fact.** *If* $\tan 3\theta$ *is defined,*

$$\tan^2\theta + \tan^2(60^\circ - \theta) + \tan^2(60^\circ + \theta) = 9\tan^2 3\theta + 6. \tag{2}$$

*Proof.* Let $x = \tan\theta$, and observe the following:

$$\begin{aligned}
&\tan^2\theta + \tan^2(60^\circ - \theta) + \tan^2(60^\circ + \theta)\\
&\quad = x^2 + \left(\frac{\sqrt{3} - x}{1 + \sqrt{3}x}\right)^2 + \left(\frac{\sqrt{3} + x}{1 - \sqrt{3}x}\right)^2\\
&\quad = x^2 + \frac{(\sqrt{3} - x)^2(1 - \sqrt{3}x)^2 + (\sqrt{3} + x)^2(1 + \sqrt{3}x)^2}{(1 - 3x^2)^2}\\
&\quad = x^2 + \frac{6x^4 + 44x^2 + 6}{(1 - 3x^2)^2}\\
&\quad = \frac{9x^6 + 45x^2 + 6}{(1 - 3x^2)^2}\\
&\quad = \frac{9(x^6 - 6x^4 + 9x^2) + 6(9x^4 - 6x^2 + 1)}{(1 - 3x^2)^2}\\
&\quad = 9\tan^2 3\theta + 6
\end{aligned}$$

by (1). So the assertion is proved.

**Examples.** Letting $\theta = 10^\circ$ and $20^\circ$ in (2), we have

$$\tan^2 10^\circ + \tan^2 50^\circ + \tan^2 70^\circ = 9\tan^2 30^\circ + 6 = 9, \tag{3}$$

$$\tan^2 20^\circ + \tan^2 40^\circ + \tan^2 80^\circ = 9\tan^2 60^\circ + 6 = 33. \tag{4}$$

Hence, combining (3) and (4), and the values $\tan 30^\circ$ and $\tan 60^\circ$, we obtain

$$\tan^2 10^\circ + \tan^2 20^\circ + \tan^2 30^\circ + \cdots + \tan^2 80^\circ = \frac{136}{3}. \tag{5}$$

If we let $\theta = 5^\circ$, $15^\circ$, and $25^\circ$ in (2), we obtain

$$\tan^2 5^\circ + \tan^2 55^\circ + \tan^2 65^\circ = 9\tan^2 15^\circ + 6 = 69 - 36\sqrt{3}, \tag{6}$$

$$\tan^2 15^\circ + \tan^2 45^\circ + \tan^2 75^\circ = 9\tan^2 45^\circ + 6 = 15, \tag{7}$$

$$\tan^2 25^\circ + \tan^2 35^\circ + \tan^2 85^\circ = 9\tan^2 75^\circ + 6 = 69 + 36\sqrt{3}. \tag{8}$$

Thus, (6), (7), and (8) yield

$$\tan^2 5^\circ + \tan^2 15^\circ + \tan^2 25^\circ + \cdots + \tan^2 85^\circ = 153. \tag{9}$$

Therefore, it follows from (5) and (9) that

$$\tan^2 5^\circ + \tan^2 10^\circ + \tan^2 15^\circ + \cdots + \tan^2 80^\circ + \tan^2 85^\circ = \frac{595}{3}.$$

**Remarks.**

(i) For $\theta = 7.5^\circ$ and $22.5^\circ$ the exact value of the left hand side of (2) is easily obtained.

(ii) From (2) we can derive a triple angle formula for cotangent:

$$\cot^2\theta + \cot^2(60^\circ - \theta) + \cot^2(60^\circ + \theta) = 9\cot^2 3\theta + 6.$$

So the formulas of (3)–(9) can be rewritten in terms of cotangent.

(iii) It is not likely to have a *simple* formula

$$\tan^2\theta + \tan^2(\phi - \theta) + \tan^2(\phi + \theta) = f(\tan 3\theta),$$

where $\phi$ is some specific angle and the right hand side is a function of $\tan 3\theta$ like the right hand side of (2).

## The Murder Mystery Method for Determining Whether a Vector Field is Conservative

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We describe here a variation of the usual procedure for determining whether a vector field is conservative and, if it is, for finding a potential function. We have used this

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