Irrational Roots of Integers

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We all know that if n is not a perfect square, then $\sqrt{n}$ is irrational. In many courses we prove this for $\sqrt{2}$ and then ask students to prove it for $\sqrt{3}$ or $\sqrt{5}$. Often this involves altering observations about even and odd integers into comments about divisibility by 3 or 5. While there is some pedagogical benefit to this, it strikes us as inefficient when there is a single concise proof that deals with all n simultaneously. The proof below resembles ones given in the literature by Bloom [**1**], Sagher [**5**], Waterhouse [**8**], and D'Angelo and West [**2**]. Bloom also cites Eves [**3**], who reports it being presented in a 1985 public lecture of Ivan Niven. Sprow [**6**] warns about Waterhouse's hidden use of unique factorization. Various other proofs appear in an article by Kalman, Mena, and Shahriari [**4**], including the following:

Theorem 1. *If $k^2 < n < (k+1)^2$ for natural numbers n and k, then $\sqrt{n}$ is irrational.*

Proof. (By contradiction.) Suppose that $\sqrt{n}$ is rational, and let it be written with least positive denominator as $\frac{a}{b}$. Now $k < \sqrt{n} = \frac{a}{b} < k+1$ implies that $0 < a - bk < b$, and so

$$\frac{a}{b} = \frac{a(a-bk)}{b(a-bk)} = \frac{a^2 - abk}{b(a-bk)}.$$

But of course $a^2 = nb^2$, so we may substitute and cancel to get

$$\frac{a}{b} = \frac{nb^2 - abk}{b(a-bk)} = \frac{nb - ak}{a - bk}.$$

This is a new rational representation of n with a smaller positive denominator, contrary to our supposition. ■

As a matter of fact, with a little extra effort, we can provide a single proof for the irrationality of $\sqrt[r]{n}$ for any integer $r \geq 2$ and any natural number n that is not an rth power. This proof resembles one by Vaughn [**7**], but we have avoided using induction.

Theorem 2. *If k, r, and n are natural numbers, and $k^r < n < (k+1)^r$, then $\sqrt[r]{n}$ is irrational.*

We use Theorem 1 only as motivation, but introduce a lemma to facilitate the proof.

Lemma.

(a) *If $x = \frac{a}{b}$ is rational and b is the minimum possible positive denominator in any rational expression for x, and if $x = \frac{c}{d}$ is another rational representation, then $b \mid d$.*

(b) *If $\sqrt[r]{n} = \frac{a}{b}$ where b is the minimum possible positive denominator in any rational expression for $\sqrt[r]{n}$, then $\sqrt[r]{n^{r-1}} = \frac{a^{r-1}}{b^{r-1}}$ and b^{r-1} is the minimum possible positive denominator in any rational expression for $\sqrt[r]{n^{r-1}}$.*

Proof of Lemma. For $x = \frac{a}{b}$, the denominator is minimum implies that a and b are relatively prime. But then $x = \frac{a}{b} = \frac{c}{d}$ implies that $ad = bc$. Since $b \mid ad$ and $\gcd(a, b) = 1$, we have $b \mid d$ as required.

For part (b) it is evident that

$$\sqrt[r]{n^{r-1}} = \frac{a^{r-1}}{b^{r-1}}.$$

If this is not the minimum denominator, then there is another fraction

$$\frac{c}{d} = \frac{a^{r-1}}{b^{r-1}}.$$

We see that $a^{r-1}d = b^{r-1}c$. Since $\gcd(a, b) = 1$, we observe that $\gcd(a^{r-1}, b^{r-1}) = 1$. (If we have the goal of avoiding the need to use prime factorization here, we can argue that $\gcd(a, b) = 1$ implies existence of an integer combination $ai + bj = 1$. Then $(ai + bj)^{2r-1} = 1$ can be expanded and regrouped to demonstrate that $\gcd(a^{r-1}, b^{r-1}) = 1$ without reference to primes.) Consequently, we have $b^{r-1} \mid d$, so in fact d cannot be a smaller denominator. ■

We now prove Theorem 2 by contradiction. Observe that the hypothesis requires that $k < \sqrt[r]{n} = \frac{a}{b} < k + 1$, which guarantees that $a - bk > 0$. We modify the fraction for $\sqrt[r]{n^{r-1}}$ as follows:

$$\begin{aligned}
\sqrt[r]{n^{r-1}} = \frac{a^{r-1}}{b^{r-1}} &= \frac{a^{r-1}(a - bk)}{b^{r-1}(a - bk)} \\
&= \frac{a^r - a^{r-1}bk}{b^{r-1}(a - bk)} \\
&= \frac{nb^r - a^{r-1}bk}{b^{r-1}(a - bk)} \\
&= \frac{nb^{r-1} - a^{r-1}k}{b^{r-2}(a - bk)}.
\end{aligned}$$

Since the minimum denominator for $\sqrt[r]{n^{r-1}}$ is b^{r-1}, we get $b^{r-2}(a - bk) \geq b^{r-1}$, which reduces to $a - bk \geq b$. Therefore, $\frac{a}{b} \geq k + 1$, which is a contradiction.

References

1. D. M. Bloom, A one-sentence proof that square root of 2 is irrational, *Math. Magazine* **68** (1995) 286.
2. J. D'Angelo, and D. West, *Mathematical Thinking: Problem Solving and Proof*, 2nd ed., Prentice Hall, 2000.
3. H. Eves, *An Introduction to the History of Mathematics*, 6th ed., W. B. Saunders, 1990.
4. D. Kalman, R. Mena, and S. Shahriari, Variations on an irrational theme—Geometry, dynamics, algebra, *Math. Magazine* **70** (1997) 93–104.
5. Y. Sagher, What Pythagoras could have done, *Amer. Math. Monthly* **95** (1988) 117.
6. D. J. Sprows, Irrationals and the fundamental theorem of arithmetic, *Amer. Math. Monthly* **96** (1989) 732.
7. H. E. Vaughan, On the irrationality of roots, *Amer. Math. Monthly* **67** (1960) 576–578.
8. W. C. Waterhouse, Why square roots are irrational, *Amer. Math. Monthly* **93** (1986) 213–214.