# CLASSROOM CAPSULES

EDITOR

*Michael K. Kinyon*
*Indiana University South Bend*
*South Bend, IN 46634*

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

## The Theorem of Cosines for Pyramids

Alexander Kheyfits (alexander.kheyfits@bcc.cuny.edu), Bronx Community College, CUNY, Bronx, NY 10453

The classical three-millennia-old Pythagorean theorem is still irresistibly attractive for lovers of mathematics—see, for example, [**1**] or a recent survey [**8**] for its numerous old and new generalizations. It can be extended to 3-space in various ways. A straightforward extension states that the square of a diagonal of a right rectangular parallelepiped is equal to the sum of squares of its adjacent edges. A less trivial generalization is the following result, which can be found in good collections of geometric problems as well as in a popular calculus textbook [**7**, p. 858]:

**(PTT) The Pythagorean theorem for tetrahedra with a trirectangular vertex.** *In a tetrahedron with a trirectangular vertex, the square of the area of the opposite face to this vertex is equal to the sum of the squares of the areas of three other faces.*

Consider a triangle with sides $a$, $b$, $c$ and the angle $C$ opposite side $c$. The classical Pythagorean theorem is a particular case of the *Theorem* or *Law of Cosines*, $c^2 = a^2 + b^2 - 2ab\cos C$, if we specify here $C$ to be a right angle. Similarly, statement **(PTT)** is a special case of the following assertion, if one puts $\theta_{12} = \theta_{13} = \theta_{23} = \pi/2$ in (1) below.

**(TCT) The Theorem of Cosines for Tetrahedra.** *In any tetrahedron with faces $f_0, f_1, f_2, f_3$, let $A_0, A_1, A_2, A_3$ be their areas and $\theta_{12}, \theta_{13}, \theta_{23}$ be the dihedral angles between, respectively, faces $f_1$ and $f_2$, $f_1$ and $f_3$, $f_2$ and $f_3$. Then*

$$A_0^2 = A_1^2 + A_2^2 + A_3^2 - 2A_1A_2\cos\theta_{12} - 2A_1A_3\cos\theta_{13} - 2A_2A_3\cos\theta_{23}. \qquad (1)$$

Hereafter, all dihedral angles between faces of a polyhedron are measured inside the polyhedron.

Being younger than the Pythagorean Theorem, this statement is just about four centuries old (Faulhaber [**3**], see [**5**, p. 1007]). As is shown below, (1) can be generalized to any pyramid.

*Proof.* The proof of (1) can be carried over by using the vector equation (see [**7**, p. 858])

$$\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0, \tag{2}$$

where the vector $\mathbf{v}_i$ is perpendicular to $f_i$, pointing outward, and such that $|\mathbf{v}_i| = A_i$. To prove (2), it is enough to express these vectors through the vectors corresponding to the three adjacent edges of the tetrahedron.

To deduce (1) from (2), it is enough to rewrite (2) as $-\mathbf{v}_0 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ and then to dot-square both sides of the latter equation, keeping in mind that the angle between $\mathbf{v}_i$ and $\mathbf{v}_j$ is supplementary to $\theta_{ij}$. ■

Since any face of a tetrahedron can be designated as its base, there are three more equations similar to (1). It is worth noting also that unlike the plane theorem of cosines above, where it is always true that $a^2 + b^2 - 2ab\cos C \geq 0$ for any real $a, b, C$, the right-hand side of (1) can be negative. To see this, it suffices to consider $A_1 = A_2 = A_3$ and $\theta_{12} = \theta_{13} = \theta_{23} < \frac{\pi}{3}$. Indeed, in a tetrahedron these parameters cannot be assigned arbitrarily, since it is known [**4**, p. 50] that in any trihedral angle it must be $\theta_{12} + \theta_{13} + \theta_{23} > \pi$.

If a plane domain can be triangulated or approximated by triangulable domains, one can use the same method to prove a slightly more general result [**6**, p. 517]:

*The sum of the squares of the areas of the projections of a plane region on the three coordinate planes equals the square of the area of the region.*

Let us return to the **(PTT)** statement and denote the lengths of the pairwise perpendicular edges of a trirectangular tetrahedron by $a, b, c$. Then the areas of the lateral faces are $\frac{1}{2}a \cdot b$, $\frac{1}{2}a \cdot c$, $\frac{1}{2}b \cdot c$, and the third edges of these faces, which are the sides of $f_0$, are $Q = \sqrt{a^2 + b^2}$, $R = \sqrt{a^2 + c^2}$, and $S = \sqrt{b^2 + c^2}$. Substituting these quantities into (1) with $\cos\theta_{12} = \cos\theta_{13} = \cos\theta_{23} = 0$, we deduce after some algebraic transformations that

$$A_0 = \sqrt{p(p - Q)(p - R)(p - S)}, \tag{3}$$

where $p = (1/2)(Q + R + S)$ is the half-perimeter of the face $f_0$. Noting that all these transformations are reversible, we see that the Pythagorean theorem for a trirectangular tetrahedron **(PTT)** is equivalent to the **Heron formula** (3) expressing the area of a plane triangle through the lengths of its sides.

Our goal in this note is to present the following generalization of (1) to arbitrary, even non-convex pyramids.

**(TCP) The Theorem of Cosines for Pyramids.** *In any pyramid with faces $f_0, f_1, f_2, \ldots, f_n$, $n \geq 3$, let $A_0, A_1, A_2, \ldots, A_n$ be their areas and let $\theta_{ij}$ be the dihedral angles between faces $f_i$ and $f_j$, $1 \leq i < j \leq n$. Then*

$$A_0^2 = A_1^2 + A_2^2 + \cdots + A_n^2 - 2\sum_{1\leq i<j\leq n} A_i A_j \cos\theta_{ij}. \tag{4}$$

*Proof.* Clearly, it suffices to extend equation (2) to this case and then to find the dot product $\mathbf{v}_0 \cdot \mathbf{v}_0$ as before. First we consider a quadrangular pyramid with a base $f_0$ and lateral faces $f_1, \ldots, f_4$. The base always, even if it is not convex, has a diagonal that dissects it into two triangles, $f_{01}$ and $f_{02}$. The areas of $f_{01}$ and $f_{02}$ add up to the area of $f_0$, so $\mathbf{v}_{01} + \mathbf{v}_{02} = \mathbf{v}_0$, where $\mathbf{v}_{0i}$ is an outward vector orthogonal to $f_i$, $i = 1, 2$, such that $|\mathbf{v}_{0i}|$ is the area of $f_i$. These triangles are bases of two disjoint tetrahedra, whose union is the given pyramid. Let $\widetilde{f}$ be the common lateral face of these tetrahedra, then the corresponding outward vectors, perpendicular to $\widetilde{f}$ in the two tetrahedra, have the same magnitude and the opposite directions. Thus, writing down equation (2) for each of these two tetrahedra and adding these equations, we arrive at the following equation for any quadrangular pyramid,

$$\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 0,$$

where as before, $\mathbf{v}_j$ is the outward vector, perpendicular to the face $f_j$, such that $|\mathbf{v}_j|$ is the area of $f_j$, $1 \leq j \leq 4$. A similar equation for any pyramid with an $n-$gonal base for any $n \geq 3$,

$$\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = 0,$$

follows by mathematical induction. ■

Since any polyhedron can be decomposed into tetrahedra [**4**, p. 67], we can use the same argument as in the proof to derive an analog of (4) for any polyhedron with any of its faces singled out as the base; the sum on the right will include all dihedral angles between all faces excluding the base. In essence this assertion is just three-centuries old and can be traced back to Carnot [**2**, p. 313].

Consider, for example, a right prism with a rectangular base. The face $\overline{f}$, parallel to the base, is perpendicular to any lateral face, so that all corresponding cosines vanish. The same holds true for each pair of adjacent lateral faces. Next, if $f'$ and $f''$ are two parallel lateral faces, then the corresponding orthogonal vectors $\mathbf{v}'$ and $\mathbf{v}''$ have opposite directions but equal magnitudes, and the dihedral angle between these two faces is zero; thus, $(A')^2 + (A'')^2 - 2A'A'' \cos\theta = 0$. The same is valid for another pair of opposite faces, and (4) reduces to a trivial equation $A_0 = \overline{A}$, that is, the opposite faces in such a prism have equal areas. A similar, but slightly more involved argument works for any prism.

It should be also mentioned that the known equations

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A,$$
$$\cos A = -\cos B\cos C + \sin B\sin C\cos\alpha,$$

where $\alpha, \beta, \gamma$ are the plane angles of a trihedral angle and $A, B, C$ are its opposite dihedral angles, are also sometimes called (the first and second) 3-dimensional cosine theorems of spherical trigonometry. We leave proofs of these equations as an exercise to the reader.

*Acknowledgment.* This work was partially supported by Bronx Community College.

## References

1. C. B. Boyer, *History of Analytic Geometry*, Scripta Mathematics, Yeshiva Univ., 1956.
2. L. N. M. Carnot, *Géométrie de position*, J. B. M. Duprat, 1803.
3. F. Faulhaber, *Miracula Arithmetica*, Augsburg, 1622.

4. J. Hadamard, *Leçons de Géométrie Élémentaire*, Vol. 2, Géométrie dans l'espace, Armand Colin, 1901.
5. J. Naas, *Mathematisches Wurterbuch*, Vol. 1, Akademie-Verlag, Berlin, 1961.
6. W. F. Osgood, W. C. Graustein, *Plane and Solid Analytic Geometry*, MacMillan, 1930.
7. J. Stewart, *Calculus*, 5th ed., Brooks/Cole, 2003.
8. D. Veljan, The 2500-year-old Pythagorean theorem, *Math. Mag.* **73** (2000), 259–272.

## Logarithmic Differentiation: Two Wrongs Make A Right

Noah Samuel Brannen (sam.brannen@sonoma.edu) and Ben Ford (ben.ford@sonoma.edu), Sonoma State University, Rohnert Park, CA 94928

You're teaching a Calculus class, and get to the point in the course where it is time to discuss logarithmic differentiation. As usual, you begin by asking the class "If $f(x) = x^x$, what is $f'(x)$?"

One student raises his hand and says "That's just the power rule. It's $xx^{x-1}$." Another student says "That's just for when the exponent is constant. This has $x$ in the exponent, so it's like $e^x$, but when it's not $e$ you need to put the log in. So it's $(\ln x)x^x$." Smiling, you point out that just as one is assuming that $n$ is a constant when one uses the formula for the derivative of $x^n$, one is assuming that $a$ is a constant when one uses the formula for the derivative of $a^x$. Since neither the base nor the exponent of $x^x$ is constant, the function $f(x) = x^x$ is neither a power function nor an exponential function, and therefore the derivative of $x^x$ cannot be found using either of these rules. Instead, you say, we will use a technique called logarithmic differentiation.

(Pedagogical aside: Of course, here you have the option of using the definition $x^x = e^{x \ln x}$; you have already covered methods for differentiating this. But the technique of using logarithmic differentiation to break the natural log of a function into a sum of easily-differentiable summands is a nice one, and this is a very good context in which to introduce it.)

At this point a student in the back of the class raises her hand and says "Isn't $f(x) = x^x$ a combination of a power function and an exponential function, and therefore shouldn't the derivative be a combination of the derivative of a power function and the derivative of an exponential function?"

Trying not to discourage the student, you attempt to take her question seriously. You ask if she means that the derivative should be the sum of the two answers given by the first two students. She replies "sure."

You decide to indulge the student, saying "Let's see what happens. The sum of the two answers is $xx^{x-1} + (\ln x)x^x$. The real answer can be found as follows: First, we let $y = x^x$. Then we take the natural logartithm of both sides, obtaining $\ln y = \ln(x^x) = x \ln x$. Differentiating both sides of this equation with respect to $x$ gives us

$$\frac{y'}{y} = (1)(\ln x) + x\left(\frac{1}{x}\right) = \ln x + 1,$$

and multiplying both sides by $y$ yields

$$y' = y(\ln x + 1).$$

Substituting $x^x$ for $y$, we see that the derivative $y'$ of $y = x^x$ is

$$y' = x^x(\ln x + 1).\text{"}$$

 