Partial Fraction Decomposition by Division

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In this note, we present a method for the partial fraction decomposition of two algebraic functions: (i) $f(x)/(ax+b)^t$ and (ii) $f(x)/(px^2+qx+r)^t$, where $f(x)$ is a polynomial of degree n, t is a positive integer, and px^2+qx+r is an irreducible ($q^2 < 4pr$) quadratic polynomial. Our algorithm is relatively simple in comparison with those given elsewhere [**1, 2, 3, 4, 5, 6, 7, 8**]. The essence of the method is to use repeated division to re-express the numerator polynomial in powers of the normalized denominator. Then upon further divisions, we obtain a sum of partial fractions in the form $A_i/(ax+b)^i$ or $(B_jx+C_j)/(px^2+qx+r)^j$ for the original function.

For (i), we let $c = b/a$, and express $f(x)$ as follows:

$$\begin{aligned} f(x) &= A_n(x+c)^n + A_{n-1}(x+c)^{n-1} + \cdots + A_2(x+c)^2 + A_1(x+c) + A_0 \\ &= (x+c)[A_n(x+c)^{n-1} + A_{n-1}(x+c)^{n-2} + \cdots + A_2(x+c) + A_1] + A_0, \end{aligned}$$

where each A_i is a real coefficient to be determined. Then the remainder after we divide $f(x)$ by $x+c$ gives the value of A_0. The quotient is

$$q_1(x) = (x+c)[A_n(x+c)^{n-2} + A_{n-1}(x+c)^{n-3} + \cdots + A_3(x+c) + A_2] + A_1.$$

If we now divide $q_1(x)$ by $x+c$, we see that the next remainder is A_1 and the quotient is

$$q_2(x) = (x+c)[A_n(x+c)^{n-3} + A_{n-1}(x+c)^{n-4} + \cdots + A_3] + A_2.$$

Continuing to divide in this manner $n-1$ times, we get the quotient $q_{n-1}(x) = A_n(x+c) + A_{n-1}$. Finally, dividing $q_{n-1}(x)$ by $x+c$, we obtain the last two coefficients, A_{n-1} and A_n. Thus, it follows that

$$\frac{f(x)}{(ax+b)^t} = \frac{1}{a^t}\left[\frac{A_n}{(x+c)^{t-n}} + \frac{A_{n-1}}{(x+c)^{t-n+1}} + \cdots + \frac{A_1}{(x+c)^{t-1}} + \frac{A_0}{(x+c)^t}\right]. \quad (1)$$

For example, to find the partial fraction decomposition of $(x^4+2x^3-x^2+5)/(2x-1)^5$, we use $c = -1/2$ and perform synthetic division to obtain A_0 through A_n.

1/2)	1	2	-1	0	5
		1/2	5/4	1/8	1/16
	1	5/2	1/4	1/8 |	81/16 $\Leftarrow A_0$
		1/2	3/2	7/8	
	1	3	7/4 |	1 $\Leftarrow A_1$	
		1/2	7/4		
	1	7/2 |	7/2 $\Leftarrow A_2$		
		1/2			
$A_4 \Rightarrow$	1	4 $\Leftarrow A_3$			

Substituting the coefficients into (1), we have

$$
\begin{aligned}
\frac{x^4+2x^3-x^2+5}{(2x-1)^5} &= \frac{1}{2^5}\left[\frac{1}{(x-1/2)}+\frac{4}{(x-1/2)^2}+\frac{7/2}{(x-1/2)^3}+\frac{1}{(x-1/2)^4}\right.\\
&\qquad \left.+\frac{81/16}{(x-1/2)^5}\right]\\
&= \frac{1/16}{2x-1}+\frac{1/2}{(2x-1)^2}+\frac{7/8}{(2x-1)^3}\\
&\qquad +\frac{1/2}{(2x-1)^4}+\frac{81/16}{(2x-1)^5}.
\end{aligned}
$$

For (ii), we let $u = q/p$, $v = r/p$, and express $f(x)$ in the following form:

$$
\begin{aligned}
f(x) &= B_{(n-1)/2}(x^2+ux+v)^{(n-1)/2} + B_{(n-3)/2}(x^2+ux+v)^{(n-3)/2} + \cdots\\
&\qquad + B_1(x^2+ux+v) + B_0,
\end{aligned}
$$

where each coefficient B_k, $k = 0, 1, \ldots, (n-1)/2$, is a linear function of x, and where we assume that $n \leq 2t-1$. In this case, dividing $f(x)$ and each successive quotient by x^2+ux+v as described above, we obtain

$$
\begin{aligned}
\frac{f(x)}{(px^2+qx+r)^t} &= \frac{1}{p^t}\left[\frac{B_{(n-1)/2}}{(x^2+ux+v)^{t-(n-1)/2}}\right.\\
&\qquad \left. + \frac{B_{(n-3)/2}}{(x^2+ux+v)^{t-(n-3)/2}} + \cdots + \frac{B_0}{(x^2+ux+v)^t}\right].
\end{aligned}
\tag{2}
$$

For instance, take the rational function $(x^5-4x^4+3x^2-2)/(x^2-x+2)^3$. Then $u=-1$ and $v=2$. Since $(n-1)/2 = 2$, we let

$$x^5-4x^4+3x^2-2 = (Mx+N)(x^2-x+2)^2 + (Kx+L)(x^2-x+2) + Ix + J.$$

Since most students are not familiar with the synthetic division technique when the divisor is a quadratic polynomial, long division can be used in place of the following computation to find the coefficients I, J, K, L, M, and N.

$$
\begin{array}{cccccc|cccc}
1 & -4 & 0 & 3 & 0 & -2 & 1 & -3 & -5 & 4\\
1 & -1 & 2 & & & & 1 & -1 & 2 & \\
\hline
 & -3 & -2 & 3 & & & & & & \\
 & -3 & 3 & 6 & & & & & & \\
\cline{2-4}
 & & -5 & 9 & 0 & & & & & \\
 & & -5 & 5 & -10 & & & & & \\
\cline{3-5}
 & & & 4 & 10 & -2 & & & & \\
 & & & 4 & -4 & 8 & & & & \\
\cline{4-6}
 & & & & 14 & -10 & & & & \\
 & & & & I & J & & & &
\end{array}
$$

$$
\begin{array}{rrrr|rrr}
 & & & & M & N & \\
1 & -3 & -5 & 4 & 1 & -2 & \\
1 & -1 & 2 & & 1 & -1 & 2 \\
\hline
 & -2 & -7 & 4 & & & \\
 & -2 & 2 & -4 & & & \\
\cline{2-4}
 & & -9 & 8 & & & \\
 & & K & L & & &
\end{array}
$$

Substituting the coefficients in (2) (note that $t - (n-1)/2 = 1$) gives

$$\frac{x^5 - 4x^4 + 3x^2 - 2}{(x^2 - x + 2)^3} = \frac{x-2}{x^2 - x + 2} + \frac{-9x + 8}{(x^2 - x + 2)^2} + \frac{14x - 10}{(x^2 - x + 2)^3}.$$

Note also that $x^2 - x + 2 = (x - 1/2)^2 + 4/7$. On the right hand side of the above expression, replacing the coefficients -2, 8, and -10 in the numerators by $-2 + 1/2M$, $8 + 1/2M$, and $-10 + 1/2M$, respectively, we get

$$\frac{x^5 - 4x^4 + 3x - 2}{((x - 1/2)^2 + 7/4)^3} = \frac{(x - 1/2) - 3/2}{(x - 1/2)^2 + 7/4} + \frac{-9(x - 1/2) + 7/4}{((x - 1/2)^2 + 7/4)^2} + \frac{14(x - 1/2)^2 - 3}{((x - 1/2)^2 + 7/4)^3}.$$

This last expression is an easily antidifferentiable form.

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An Elegant Mode for Determining the Mode

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For any probability distribution, the mode, like the mean and median, is a measure of central tendency. Geometrically, it represents the relative maximum of the probability density function (pdf) and thus is the most striking feature in the curve's topogra-