CLASSROOM CAPSULES

EDITOR

Michael K. Kinyon
Indiana University South Bend
South Bend, IN 46634

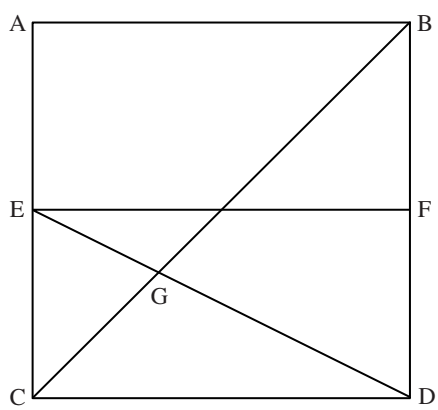
Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the new Editors Ricardo Alfaro and Steven Althoen, University of Michigan-Flint, Flint, MI 48502.

Revisited: $\arctan 1 + \arctan 2 + \arctan 3 = \pi$

Michael W. Ecker (DrMWEcker@aol.com), Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA 18627

Here's a neat challenge: Given a square piece of paper, fold the paper as few times as possible so as to create angles whose measures are arctan 1, arctan 2, and arctan 3. (A fold is counted even if its only purpose is to locate a point.) Can we use only the geometry of the construction to prove the identity in the title? We can if we show that all three arctangents are angles of one triangle.

Solution: Three folds suffice for the construction.



1 Fold along diagonal BC.

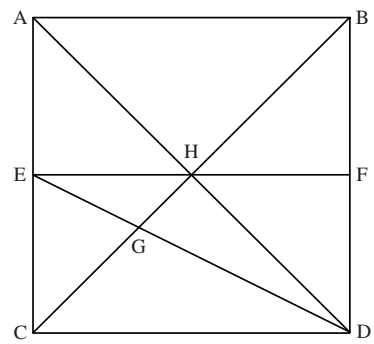
#2 Bring AB to CD and fold along EF. This locates E.

#3 Fold along DE, meeting BC at G.

From the diagram, angle ECG has measure arctan 1 and angle CEG measures arctan 2. However, we've used up our three folds. Can we show angle CGE measures arctan 3?

Add one more labeled point, below: Let H be the intersection of EF and BC. As H is also the intersection of AD and BC, it is the center of the square. Note that H is the midpoint of AD (AH = DH) and E is the midpoint of AC (AE = CE), making CH and DE medians of triangle ACD. By a well-known property of medians, GH must

be one-third the length of CH. As CH = DH, GH is also one-third the length of DH. Note that angle DHG is a right angle and also that vertical angles DGH and CGE are equal in measure. Hence, from those facts and right triangle DHG, the tangent of angle CGE = the tangent of angle DGH = DH/GH = 3. ■



Acknowledgment. Thanks to Harry Baldwin of San Diego, Calif. for fruitful correspondence on this.

A Bug Problem

Aaron Melman (amelman@scu.edu), Santa Clara University, Santa Clara, CA 95053

Imagine a vessel, obtained by revolving the function $y = x^2$ around the y-axis. On the wall on the inside of this vessel sits a bug, which becomes fairly unhappy when a liquid of your choice is poured into the vessel at a constant rate of ρ liters/second. Naturally, the bug will crawl upward to avoid getting its feet wet. If it is crawling along the curve $y = x^2$ in the xy-plane, how fast does it have to crawl to outrun the rising tide of the liquid? The vessel may be considered to be as large as necessary.

The solution to this problem is not difficult, but it requires familiarity with volumes, arc length, the fundamental theorem of calculus, and the chain rule. As such, it is a good review problem for a calculus class. Moreover, the problem can be generalized easily enough for students to explore.

To solve the problem, let us first express the volume of the liquid as a function of time. Since the flow rate is constant, this is simply $V(t) = \rho t$, where we have set the initial time $t_0 = 0$. To find the height in the vessel that this volume corresponds to, we compute the volume $\phi(h)$ of a vessel of height h. An easy way to do this is to use the inverse function $x = \sqrt{y}$ and circular cross-sections. The result is $\phi(h) = \int_0^h \pi(\sqrt{y})^2\,dy = \pi h^2/2$. Therefore, the height of the rising liquid is determined by $\rho t = \pi h^2/2$, or $h(t) = \sqrt{2\rho t/\pi}$. On the other hand, the bug is crawling along the curve $x = \sqrt{y}$, and, assuming that it starts from a height $h_0 > 0$, the distance it covers is given by

$$L(t) = \int_{h_0}^{h(t)} \sqrt{1 + ((\sqrt{y})')^2}\,dy = \int_{h_0}^{h(t)} \sqrt{1 + \frac{1}{4y}}\,dy.$$