MATHEMATICS IN SCHOOL AND COLLEGE*

Editorial Note: The following article is an extract from the book *General Education in School and College*, Harvard University Press, 1952, pp. 52–57. This book is the outcome of a study designed to improve the integration between school and college undertaken by members of the faculties of Andover, Exeter, Lawrenceville, Harvard, Princeton, and Yale. The portion of the report dealing with mathematics was based, in part, on a survey conducted by Richard S. Pieters (Andover) and on a panel discussion in which the committee was joined by A. W. Tucker (Princeton), E. G. Begle (Yale), Peter J. Kiernan (Lawrenceville), Ransom Lynch (Exeter), Winfield M. Sides and Richard S. Pieters (Andover).

A reading of the entire report reveals that the recommendations below were framed with special reference to the better-than-average student. Also it should be noted that the conclusions are intended to apply to the relationships between the six institutions represented and are not necessarily valid in other situations. Your editor believes, however, that the basic ideas expressed can be of great value to mathematicians in other schools and universities who may be interested in incorporating suitable modifications of these suggestions into their own curricula.

No subject is more properly a major part of secondary education than mathematics. None has a more distinguished history or a finer tradition of teaching. Perhaps the very excellence of the topic has helped, in recent decades, to keep the content and order of its teaching largely unexamined. One of the most remarkable of our sessions was the one in which we consulted with a group of first-rate school and college teachers of mathematics and discovered, as the evening progressed, a very high degree of consensus on the view that school offerings in mathematics are ready for drastic alteration and improvement.

At present the basic four-year course of mathematics in the schools we have studied covers two years of algebra, one of plane geometry, and one of solid geometry and trigonometry (the latter year elective). A small group of boys (less than one in five) are advanced more rapidly in two of the schools so that in their last year they are introduced to the calculus in a course roughly comparable to first-year college calculus. This basic four-year curriculum has the sanction of the ages, and there can be no doubt that every subject in it has a value of its own. But the Committee was still more impressed by the value of what is usually omitted, and by the fact that in some cases the words of principle seem to be obscured by the trees of constant repetition and problem-solving.

Mathematics is the rigorous application of notions that are vaguely apparent even to the nonmathematical; its wonder lies in its demonstration of the extraordinary power which comes from thinking closely and connectedly. It is not remarkable that an apple drops with growing speed from tree to ground, but what Newton did to this idea is one of the heroic advances of the human mind. The means by which he did it are not easily understood; the power of abstract thinking is matched by its difficulty. This is true for all the basic ideas of mathematics, yet it is just these basic ideas that are of the greatest value to the student. As the Harvard Report has noted, "It is unfortunately true that those aspects of algebra and geometry that are of greatest interest in general education are also more difficult to teach, and are much harder for the student to grasp, than are the technical skills of mathematical manipulation."* From every point of view—that of the college mathematics teacher as well as that of general education—we are persuaded that the greatest business of the school's mathematics curriculum should be to communicate as many of these central concepts as it can; we think the better student can do much more of this than he has been doing, and we are certain that he should try.

But from the point of view of a crowded curriculum we are convinced that each branch of mathematics operates under a law of rapidly diminishing returns. Once the basic notions are solidly understood—and this will require drill as well as thought—we think it is unwise to linger in loving elaboration of a set of ideas grown familiar. Of course it is possible to design problems of bewildering complexity in every subject from long division to trigonometry; it is also a waste of time. We do not press the paradox, but we suggest that it is almost true that the better the student, the fewer problems he should be asked to solve.

The impact of these notions upon the present mathematical curriculum is heavy, for large parts of it turn out to be relatively unhelpful elaborations of principles which are better taught in other ways. The greatest single offender in this sense is solid geometry. It is a beautiful subject, but in the strictly mathematical sense it is an elaboration of plane geometry, and elaboration is not the point of mathematics. The real value of solid geometry lies outside its mathematics, in the fact that it tends to develop a general sense of spatial reality. This can be done more briefly and more effectively, we think, if the effort to develop a systematic structure is abandoned. If generally adopted, this single revision would save nearly half a year in the standard school curriculum.

The example of solid geometry can be repeated, on a smaller scale, in the teaching of algebra, plane geometry, and trigonometry. Among the topics which seem appropriate for condensation or omission are complex numbers,† determinants, logarithmic solutions of triangles, and the geometry of the circle. These are suggestions only; much planning and experience will be needed before teachers of mathematics can determine where to modify and revise their present practice, and it is by no means certain that the problem has a unique solution. All that we assert, and we do so with the authority of our professional advisers, is that much can be squeezed out, to the positive advantage of the basic notions which are now taught.

If we are correct in claiming that there is excess fat on the body mathematical, it is good, in itself, to trim the present curriculum; for it is one of the basic conclusions of our whole study that it is bad to do in two terms what should be

* General Education in a Free Society (Harvard University Press, 1945), p. 163.

† Professor Tucker has suggested to the editor that this reference to complex numbers may have slipped in by accident or in any event was not sufficiently considered by the committee which prepared the report.

done in one. But we are not urging a cut in the traditional mathematics course because we want less time for mathematics. What we are trying to do is at once to bring into visible relief the great ideas of traditional algebra and geometry, and at the same time to make room for two additional sets of notions—those related to the calculus and to statistics. This may be too much to ask, but before we consider the difficulties let us briefly urge the importance of each of these two topics.

The importance of the calculus for all scientists and engineers needs no argument. Its respectability as a discipline in liberal education is equally plain; it is the standard freshman course in the best of our colleges. The only thing new in our position is the suggestion that the elements of this subject can be presented to a large percentage of well-trained schoolboys in the 12th grade. Not very many, perhaps, can meet the standards of a college course, but that is not our first object. What we are eager to do is to get into the minds of the student who is *not* headed for a career in science or mathematics some sense of the power and meaning of the calculus. We are firmly persuaded that this can be done in school, and it seems to us to have a value, in general education, that is greater by several orders of magnitude than the value of drill in the elaboration of solid geometry or determinants. The student who has once grasped the meaning of differentiation and integration sees the world afterward in a larger and more significant way; his exposure to science takes a different shape; his sense of space and motion is enlarged. Obviously not all students take this meaning from a single course in the calculus; not all teachers try to teach it. But for those who can, the opportunity is too great to be sacrificed to anything but necessity, and we are bound to conclude that tradition, not necessity, is what currently limits the teaching of elementary calculus in school.

There is a further and most practical advantage in an expanded teaching of the calculus in school. It is that a student of physics, however good his training in school, finds it extremely difficult to handle advanced physics courses unless he has been introduced to the calculus. The prospective scientist can clear his way in both physics and mathematics if he does solid work in the calculus in school.

In principle, the nonscientist with school training in the calculus might reap a parallel advantage; in his study of science in college he could go much further into the meaning of physics than would otherwise be possible. It is a serious misunderstanding to suppose that general education and mathematical physics are opposites. The best of the college courses for the nonscientist make considerable demands on the mathematical equipment of their students, and their teachers would be happy to build on a knowledge of elementary calculus. But such a development may take time.

To some of our consultants, and in some respects to our Committee as well, the case for statistics is even more powerful than the case for the calculus. The notions of probability, correlation, and sampling are among the fundamentals of modern social measurement. And since we live in the age of polls, an awareness of the real meaning of these notions is a protection to the consumer as well as a necessity for the producer of information. Moreover, there is in all statistics a salutary concern for the uncertain and the incomplete—for the gray that is real more than for the black and white that is abstraction. It is well for the student to learn both that mathematics has uncertainty and that uncertainty can be mathematically treated. This knowledge is important in many fields; teachers of science and teachers of history alike have their troubles with students who are persuaded that all reasoning is geometrical and all evidence conclusive. All in all, if we had a curriculum to build from the ground up, we cannot suppose that it would omit statistics from a general education.

If we could find the time, we would urge that the standard school curriculum include about one year of the calculus (with a minimal framework of analytic geometry) and half a year of statistics. And if all else were equal, we would urge that the statistics follow the calculus; it has much more meaning and power that way. But time is not easy to find, and all other things are not equal. The teaching of statistics in school can hardly grow very fast until there is a body of teachers with experience in the subject and some core of knowledge on which to build the right kind of course. Moreover, there is a real advantage in leaving the whole of the 12th grade for the calculus—this is the topic that leads directly to advanced work in college, and it is a well-tested and coherent unit which is best taught in a single academic year. Finally, nearly all good school departments of mathematics are well-equipped to teach the calculus; they have been teaching it, on a limited scale, for many years. On balance, therefore, we recommend that the schools should move toward a curriculum in which the basic 12th grade course is the introduction to the calculus. At the same time we hope that there will be intensive experimentation with the teaching of some of the basic concepts of statistics, and we think there is room for this in the second year's study of algebra. Nor do we exclude the possibility that some schools may wish to offer statistics as an alternative to the calculus, or even as an additional elective.

It is our conclusion that the school mathematics curriculum can and should be redesigned to include new areas of instruction; and we think that when this has been done, college mathematics should be a subject for scientists, mathematicians, and really talented amateurs of the topic. For others, there is plenty in the basic course we have outlined, and their college work should be in other fields. And this we feel is as it should be; fundamental mathematics of the sort we have been discussing is taught better—and learned better—in the schools than in the colleges.