both C_1 and C_2 carries the Barbilian perpendicular to both lens arcs. This, with the previous remarks, shows that the two conjugate systems of circles, defined by the base points m and n, form a gridwork of lines within the lens. It is also clear that inversion with respect to any circle which is orthogonal to both C_1 and C_2 is a motion of the space.

MATHEMATICS IN THE SECONDARY SCHOOLS FOR THE EXCEPTIONAL STUDENT‡

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1. Introduction. The work to be described in this paper was done under the auspices of the School and College Study of Admission with Advanced Standing. This Study, which was initiated by President Gordon Chalmers of Kenyon College, was undertaken by twelve colleges* and twenty-seven secondary schools. Financial support for the Study was given by The Fund for the Advancement of Education, and Dr. William H. Cornog, President of the Central High School, Philadelphia, Pennsylvania, served as its Executive Director. The fundamental principle behind the study is that certain bright students are wasting time in the schools as set up at present, and that they could profitably spend some time in anticipating the work they would normally do in their first year of college. It was proposed that the twelve colleges should reach agreement on the content of the central subjects of the college freshman year, with a view to granting advanced credit in these subjects to those students who qualified in them. It was inevitable that mathematics be one of these subjects.† A committee was appointed to deal with each of these subjects and to define the subject matter concerned. This was done in consultation with representatives of the various institutions, and many lengthy meetings were held by each committee throughout the year to discuss the problems raised by the study. The committee for mathematics consisted of the following: Julius Hlavaty (Bronx High School of Science, New York City); Elsie Parker Johnson (Oak Park and River Forest High School, Oak Park, Ill.); Charles Mergendahl (Newton High School, Newtonville, Mass.); George B. Thomas (Massachusetts Institute of Technology);

‡ Presented to the Mathematical Association of America on December 31, 1953.

* Bowdoin College, Brown University, Carleton College, Haverford College, Kenyon College, Massachusetts Institute of Technology, Middlebury College, Oberlin College, Swarthmore College, Wabash College, Wesleyan University, Williams College.

† The other subjects were: English Composition, Literature, Physics, Chemistry, Biology, Latin, French, German, Spanish, and History.

Elbridge P. Vance (Oberlin College); Volney H. Wells (Williams College), with the writer (Swarthmore College) as chairman. The reports of all the committees are available at the office of Dr. Cornog, the Executive Director of the Study.

It should be mentioned here that the twelve colleges have all adopted the proposed plan to grant advanced credit for the work as outlined by the committees. Similar committees are now at work constructing examinations, which will be used to test the candidates for such credit. Furthermore, seven specially selected schools* are testing out these ideas by preparing candidates in several of the subjects mentioned. Many of the other cooperating schools are also giving courses of the sort proposed by our committees.

The problem before our committee was to define the work in mathematics which would be acceptable by our twelve institutions for college credit in place of a first-year college course. On the basis of consultation with the departments of mathematics of the twelve institutions, our committee decided that such a first-year course should consist essentially of a substantial course in calculus with applications, along with the analytic geometry that is needed for such a course. Our committee did not want to recommend that advanced credit be given for courses in college algebra, solid geometry, and the like, even though such courses are occasionally given in the freshman year at some colleges. When we attempted to integrate our freshman course into the High School program we found that it was necessary to give consideration to the whole program in mathematics in the secondary schools. We accordingly worked out a detailed program in mathematics for the last three years of secondary school, which culminates in a course that will be acceptable for advanced standing in college. When we got through with this we found that such a program is preferable in many ways to the standard program in mathematics as it is now pursued in most secondary schools, although we started out by devising it for the exceptional student.

2. The problem. It has been realized for many years that the mathematics curriculum in the secondary schools was in for a drastic revision. For example, the section on mathematics in *General education in school and college* (reprinted in this MONTHLY, vol. 60, 1953, pages 380–383) deals with this topic, and certain general proposals for revising the curriculum are made there. It will be seen that the program planned by our committee agrees in many ways with these proposals; in addition, we have given detailed suggestions as to how such a program can be carried out.

The main ideas behind our plan are: to break down the standard compartmentalized program for these years, to introduce certain new subject matter, and at the same time to suggest the elimination of certain traditional items. As

* Bronx High School of Science (New York City); Central High School (Philadelphia, Pa.); Evanston Township High School (Evanston, Ill.); Germantown Friends School (Philadelphia, Pa.); Horace Mann School (New York City); Newton High School (Newtonville, Mass.); St. Louis Country Day School (St. Louis, Mo.).

a background for this proposed program we assumed a course of instruction carrying the student through the ninth grade; he would thus be acquainted with the number system, the vocabulary and ideas of elementary, intuitive geometry, the beginnings of the use of algebraic symbolism, and the ideas of graphical representation. Our program for the three years of senior High School would then be the following:

(1) A course in 10th year mathematics stressing deductive thinking, but dealing with various types of mathematical subject matter.

(2) A year's course consisting of a continuation of algebra, of analytic geometry, and of trigonometry.

(3) A year's course made up of certain advanced topics in algebra and analytic geometry, with a substantial introduction to calculus and its applications.

3. Tenth Year Mathematics. The work in this year has traditionally dealt with deductive geometry. Our proposal would continue to stress the deductive method, but would apply it not merely to geometrical subject matter, but to material from algebra and other subjects as well. To achieve this it is not necessary that all of the geometric content of the course be organized into one logical sequence; the role of undefined terms, definitions, assumptions, theorems, can be taught by exhibiting several instances of short groups of propositions. Each such group would illustrate the meaning of a deductive system. Furthermore, it is desirable that non-geometric material—for example, from algebra and other subjects—be organized in this fashion. Our report gives some detailed examples along these lines. In addition to the geometric content of the year's work—and we would urge that the simpler concepts of three dimensional geometry be considered along with their plane counterparts—it is essential that the study and development of the algebra begun in the 9th grade be continued here. This is best done by introducing the subject of analytic geometry at this time and carrying it to a point where the student is able to prove simple theorems by algebraic methods. The equations of simple curves (circle, parabola) can also be introduced. The student will thus continue his use of algebra and at the same time increase his geometric insight. In this connection the study of trigonometry should also be begun.

4. Eleventh Year Mathematics. The work of this year should serve as an introduction to analysis. Since the students taking mathematics at this level are generally those who are planning to go to college, the material should be of a college preparatory character. The subject matter is a continuation of work begun at earlier levels and is essentially algebra, analytic geometry, and trigonometry. Among the subjects to be treated in algebra are the theory of polynomials—including the remainder theorem; systems of linear equations—including determinants; complex numbers; logarithms. The work in analytic geometry should carry the student through the study of linear geometry and the elements of conic sections. The trigonometry to be studied in this year should be largely analytic trigonometry, the work being centered around a study of the trigonometric functions as functions, with a minimum of work done in geometric trigonometry and computation. Moreover, the application of trigonometric ideas to complex numbers should certainly be included here.

5. Twelfth Year Mathematics. The primary objective of the work in this year is to give a substantial introduction to differential and integral calculus, with enough applications to bring out the meaning and to illustrate the fundamental importance of this subject. The work to be done in the two previous years was planned so as to lead up to this subject, and the necessary prerequisite topics from algebra, analytic geometry, and trigonometry were included. It is expected that skills in these subjects will be further developed when they are employed in the calculus, and that those topics which were not covered during previous years will be studied now. In particular, the analytic geometry previously studied would be used here and would be enriched by applications of the calculus.

6. Remarks. It will be seen that there is nothing very unconventional about the subject matter suggested in our program. In fact, it is quite definitely built around the development and applications of analysis. It seems to the writer that this is the way it should be, especially when one keeps in mind the variety of students for whom this work is planned. A student entering college prepared with such a program will have many advantages. He will in fact be ready to take the normal sophomore course in mathematics. Thus, if he becomes a mathematician he will be able to accelerate his progress in analysis and will, because of this fact, be able to take additional work in other fields, such as higher algebra, advanced geometry, statistics, mechanics, and so forth, and thus round out his mathematical background. If the student is preparing to study science or engineering, he will be greatly helped by having had a course in calculus before he enters college; his first course in physics, for example, can then be a real meaningful introduction to the subject. Finally, there will be certain students who will not need to study mathematics in college, because the course here outlined will give them the kind of preparation that is sufficient for their needs. Such students will thus be free to take work in college which will perhaps be more to their advantage educationally.

As stated at the beginning, the program here outlined was originally proposed by our committee for those exceptional students who wish to present a certain amount of mathematics for advanced credit. It seems to us, however, that the ideas behind it could be equally well applied to secondary school mathematics in general. Thus it may be that work along these lines will become the standard program in secondary school mathematics. Or, it may be desirable for the less able student to stretch out over three years the work that we have laid out for the tenth and eleventh years. As a third possibility, a student may wish to take the tenth and eleventh year courses proposed in our program, and not take the twelfth year course; courses in advanced algebra, statistics, solid geometry, and so forth could be made available for such a student in his twelfth year. Such students would then take the normal freshman college course when they enter college. There are already many schools in this country in which work of this type in mathematics is being done. It is our hope that this will be continued and that other schools will follow.

ALTERNATIVE SOLUTION TO THE EHRENFEST PROBLEM*

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1. Introduction. Explicit expressions for the probabilities connected with the so-called Ehrenfest model (see Section 2 below) have been obtained by M. Kac [1], who has applied the usual method of dealing with problems involving discrete Markov chains. It is the purpose of this article to show that the solution to the problem can be obtained in a simpler way if the problem is formulated in terms of a direct product representation.† Such a formulation should be useful for calculating probabilities connected with discrete Markov chains of similar complexity.

2. The Ehrenfest model. In order to illustrate certain features of statistical mechanics, P. and T. Ehrenfest [2] have considered the following simple model.

$2R$ labelled balls are placed in 2 boxes. We have $2R$ labelled tickets, a one-to-one correspondence existing between the tickets and balls. A ticket is drawn at random. The ball represented by this ticket is removed from the box it is in and placed in the other box. The ticket is replaced in the pack which is then shuffled. The process is repeated.

One of the questions which arose in connection with this model is—what is the probability, $P(n|m; s)$, that there are $R+m$ balls in box 1 after s draws if there are $R+n$ balls in box 1 initially? This is the problem treated here.

3. Procedure. A brief outline of the procedure for calculating P follows. A set of orthonormal column vectors, ξ_i, is found such that each ξ_i represents a possible state of the system. (We shall define a state explicitly below.) A matrix operator H is then found such that when it operates on a state vector, ξ_i, it forms the sum of all those state vectors which can result from ξ_i in a single draw, *i.e.*,

* I wish to thank Professor W. Opechowski for his interest in this problem. I should also like to express my indebtedness to the National Research Council of Canada for a Studentship.

† The referee has kindly brought to my attention the existence of a paper by A. J. F. Siegert [4], in which essentially the same method is used to obtain the solution to the Ehrenfest problem. However, the present paper differs considerably from that of Siegert's in the explicit formulation of the method. It may be mentioned that Siegert has investigated other problems for which the same method of solution applies.