FRESHMAN MATHEMATICS AS AN INTEGRAL PART OF WESTERN CULTURE

MORRIS KLINE, New York University

1. Introduction. I would like to re-open the very hackneyed question of what to teach the liberal arts student. My excuse for doing this is that the answer which has been given for generations is equally hackneyed and, what is more to the point, a totally unsatisfactory one.

Let me state at the outset that by the liberal arts student I mean the one who does not intend to use mathematics in some profession or career, and is taking mathematics because the subject is supposed to contribute to a liberal education. Though much might be said about the proper freshman courses for students who intend to continue their mathematical training beyond this first year or who may take just one year of mathematics but will have to apply it in specialized physics, chemistry, and biology courses (*e.g.*, pre-medical students), the needs of these latter two groups of students will not be discussed here. The recommendations to be made in this article concerning the liberal arts students are based on the assumption that this group can be segregated from the others. This segregation can be effected even in small colleges without creating any serious administrative problems.

As defined above, the liberal arts students constitute only one group of freshmen. However, by far the greatest percentage of freshmen belongs to this group. Also, since the segregation that is presupposed is in accordance with interests rather than ability, the liberal arts group will contain some of the most worthwhile students. For these reasons, then, this group must be given the utmost consideration. Moreover, since many of these students will become leaders in our society they will determine the fate of mathematics in some areas. Hence there are selfish reasons too for being concerned about the knowledge and impressions of mathematics which these students will acquire.

2. Objections to the traditional courses. What have we been feeding the liberal arts students? The almost universal diet has been college algebra and trigonometry. I believe that these courses are a complete waste of time. What educational values are there really in exponents, radicals, logarithms, Horner's method, partial fractions, binomial theorem, the trigonometric identities, and the law of tangents, just to mention some of the conventional topics?

Let us, in fact, examine this material in terms of the values commonly claimed as warranting its inclusion in the curriculum. One hears most often that students learn to reason by studying this material. Leaving aside the question of transfer of training, a moot point at best, I maintain that one cannot teach reasoning with this material because the material does not permit it. Every one knows that it is impossible to give freshmen a satisfactory definition of an irrational number. Hence we not only beg many questions of logic in teaching operations with irrational numbers but we compound confusion by using irrational numbers as exponents in the subject of logarithms. Of course similar difficulties are ignored in the use of irrational numbers as trigonometric ratios associated with angles. In college algebra we use many properties of continuous functions without proof, relying upon a graph to convince the student of the correctness of these properties. For example, we use the fact that if the polynomial $f(x)$ has opposite signs for $x=a$ and $x=b$ then $f(x)=0$ has a root for a value of x between a and b. And we use the fact that $f(x)=0$ has at least one root. Even where we present proofs very few students can follow some of them. For example, how many students follow the proof of the binomial theorem for positive integral exponents, let alone fractions? When we teach partial fractions we generally omit the proofs because so much time would be required to present them correctly. Likewise when we teach the addition formulas and other identities of trigonometry we do not present proofs for all values of the variables because these proofs would take an unconscionable amount of time. To sum up, in our presentations of college algebra and trigonometry we mix proofs, intuitive arguments, graphical evidence, and assertions accepted without proof. Nevertheless, we expect the student to learn all the material. How he can come through with an appreciation of and respect for proof as exemplified by this one year of mathematics is not at all clear to me. I would say that we have confused him.

A second argument for mathematics is its beauty. There is beauty in mathematics. However, the question of what is beautiful must be answered subjectively. I contend that if I were to select portions of mathematics that are beautiful, they would not come from college algebra and trigonometry. As a matter of fact we know that these subjects are taught not because they are beautiful but because they are the next step towards higher mathematics after the high school subjects. And even if portions, at least, of college algebra and trigonometry were beautiful, it would be most difficult to sell mathematical beauty to students who are psychologically indisposed. To sell this beauty is like trying to give away ten dollar gold pieces. People are too suspicious even to examine one.

A third argument presented for the usual material is that everybody makes some use of mathematics and hence must learn some of it. But who of us has used college algebra and trigonometry outside of his professional work? There is actually nothing in these subjects which is ever used by an educated layman. Of course there are significant applications of college algebra and trigonometry but not only do these have no bearing on daily life but they occur in the context of advanced physics and mathematics so that the student cannot see where these techniques are of use even to scientists and engineers.

The truth about college algebra and trigonometry is that these subjects comprise nothing but a series of dry, boring, unmotivated, disconnected, and, to the student, unimportant techniques. The subjects are taught as techniques and the students are expected to master and reproduce them in parrot-like fashion. The entire body of material is of no value to the non-specialist and no argument for it, such as mental discipline, will ever make such material palatable or pregnant with significance. The only argument in favor of this material is that it is readily forgotten.

There is unfortunately too much evidence to support the above evaluation of college algebra and trigonometry as courses for the liberal arts students. Those of us who have taught college freshmen know how little they get from the material. We know that we have failed to reach them and that all we succeed in doing is to intensify their dislike of mathematics. We know that they leave our courses grateful only for the fact that the year is over and vowing never again to become involved with mathematics.

Indirectly and unintentionally many of us have supplied evidence that the above evaluation is correct. A recent advertisement for one of the new college algebra texts described it as a real innovation, "a humanized approach to mathematical concepts." Algebra was supposedly presented as a language, as a logical science, and as a branch of human endeavor. It was also described as a collection of techniques and as a collection of puzzles. I could readily appreciate these last two values but was curious as to how the others might be incorporated. I therefore made haste to examine this "novel" algebra text and found exactly two and one-half pages out of more than 400 devoted to these values of algebra. There was in fact exactly one paragraph on algebra as a language, one on algebra as a logical science, and one on algebra as a branch of human endeavor. The rest of the book was the same as any one of the $4097\frac{1}{2}$ college algebras which have appeared since 1900. The point of this illustration is not that the author failed to present the values of algebra claimed in the advertisement, though, of course, he did fail. Rather, here is an author who sees the need to present these values of mathematics, who tries to do so through the medium of college algebra, and who succeeds only in showing that the material does not lend itself to the exposition of these values.

Other authors, presumably equally dissatisfied with the meaningless and poverty stricken material of college algebra and trigonometry, add historical notes. After a long chapter on the theory of equations they tell us that Descartes was born in 1596, that he died in 1650, and that he was a philosopher. Such feeble, ridiculous, and pediculous efforts to incorporate human values are more to be pitied than scorned.

By persisting in the teaching of college algebra and trigonometry to the liberal arts student, college teachers of mathematics have been committing a heinous crime against mathematics and humanity. We have been guilty of teaching brick laying instead of architecture, and color mixing instead of painting. Such teaching has discouraged latent interest in mathematics and has embittered many young people towards mathematics, and in many cases, towards all learning. More than that, it has cut the ground from underneath the subject so that it is gradually losing its place in the liberal arts curriculum. Apropos of the teaching of the conventional material is the remark Mephistopheles makes to Faust in Goethe's drama:

Is it life, I ask, is it even prudence,
To bore thyself and bore the students?

To this quotation I would add that even the devil speaks the truth sometimes.

3. Some recent innovations in freshman mathematics courses. In recent years serious attempts have been made to improve the standard offering and I should like to discuss some of these briefly. A reasonably common alternative to college algebra and trigonometry is the mathematical analysis or general mathematics course which selects material from college algebra, trigonometry, co-ordinate geometry, and the calculus. Though the attempt to give the student a somewhat broader picture of elementary mathematics can be commended, the contents of the texts that have been published cannot be, for in effect all that has been accomplished is to select techniques from four fields instead of two. Since the basic ideas underlying the fields from which the techniques are drawn are now more numerous and in the case of the calculus, more difficult, the student is left even more at sea as to what is accomplished by all these techniques. Rigor is hardly even envisaged as an objective in these courses.

Some teachers, who are quite willing to break sharply with traditional material, attempt to teach the basic and broad concepts of modern mathematics and hence concentrate on such topics as functionality, transformation and invariance, and groups, rings, and fields. These topics are indeed basic ones in mathematics and they do constitute some of the most beautiful themes. However they are far too sophisticated and abstract to mean anything to freshmen and their beauty makes no appeal to the uninitiated. Moreover, one could hardly say that these topics present a rounded view of the multifaceted role of mathematics in our civilization.

Other teachers, highly conscious of the absence of rigor in conventional college algebra and trigonometry courses, have attempted to supply it by teaching the foundations of the real and complex number systems. This material is over the heads of freshmen. Real numbers are so familiar to them that the problem of defining them and rigorously establishing their properties does not strike them as significant. Such courses must devote a great deal of time convincing students that they ought to learn what the entire mathematical world did not miss for thousands of years. Moreover, aside from the difficulties in a really rigorous approach to the irrational number, the material is not representative either of pure or applied mathematics.

To my mind some better efforts to meet the problem presented by the liberal arts student were made by Dresden of Swarthmore in his *Invitation to Mathematics* and—if I may so speak of a contribution in which I was personally involved—by Cooley, Gans, Kline, and Wahlert of New York University in their *Introduction to Mathematics*. These two books are by no means alike. Yet both indicated a willingness to break away from conventional patterns and to abandon techniques in favor of ideas. They differ in that Dresden's book is devoted to a wide range of ideas of pure mathematics and to the presentation of rigorous proofs, whereas the second one is more descriptive, includes some applications of mathematics, and points out the significance of some of the creations in our culture. Because the style and level of Dresden's book were difficult for freshmen it did not meet wide approval. The *Introduction to Mathematics* did receive a very encouraging response as measured in terms of the number of adoptions and did stimulate other authors to write similar books. (One author liked the idea of this book so much that he took over the entire pattern and used many sections verbatim, acknowledging, of course, the inspiration of Euclid, Newton, and Einstein.) Both of these books were a step in the right direction. However, to my mind both still compromised too much with conventional patterns and failed to present adequately the significance of mathematics in our civilization.

4. The philosophy underlying the cultural approach to mathematics. Having criticized past efforts so strongly I know that the time has come to state what I think should be done. This I propose to do next. First, however, I should like to state the philosophy which guides what I propose. Actually the conviction that a particular course seems right comes first and one invents the philosophy afterwards to rationalize the conviction. Nevertheless, a wise philosophy argues eloquently for the course which abides by it.

My philosophy contains three principles. The first of these states that *knowledge is a whole* and that mathematics is a part of that whole. However the whole is not the sum of its parts. The present procedure is to teach mathematics as a subject unto itself and somehow expect the student who takes only one year of the subject to see its importance and significance for the general body of knowledge. This is like giving the student incomplete pieces of a very complicated jig-saw puzzle and expecting him to put the puzzle together. It follows from this principle that mathematics must be taught in the context of human knowledge and culture.

The principle I have stated is not at all unfamiliar or unrecognized by teachers. But either because they themselves are ignorant of the true place of mathematics in our civilization and culture or because they become too steeped in their own specialty they ignore the broad picture and expect the student to supply it.

My second principle is that *mathematics must contribute to the objectives of a liberal arts program*. Though it is difficult to state these objectives precisely one might say that they are to acclimate the student to the civilization and culture in which he lives, to increase his appreciation of what that culture contains, and to prepare him for life in his cultural environment. It follows from this principle as well as the first one that a mathematics course for liberal arts students must relate the subject to other branches of our culture.

Many teachers agree with this principle but conform to it in strange ways. They teach how to mix x pounds of coffee at 20¢ per pound with y pounds of tea at 50¢ per pound to make 100 pounds of some unspeakable mixture. In very modern textbooks this application has been made more realistic. Coffee is one dollar per pound and tea, two dollars.

My third principle is a negative one but it seems to be necessary to include it. *Don't compromise with your objectives.* Choose material and a presentation which directly fulfill the purposes of your course. Don't let tradition dictate subject matter. Don't drag the important ideas in by the tail while actually stressing conventional techniques.

5. Sketch of the proposed course. I believe the above principles and objectives can be adhered to in a satisfactory freshman liberal arts course and I propose to sketch the outlines of such a course. Briefly described, the object of the course is to present mathematics as a major constituent of modern culture. The course treats significant ideas of mathematics and the influences of these creations on other branches of our culture.

The material is arranged in historical order. The purpose is not to present the history of mathematics. Rather, the historical order is roughly the logical order. Moreover, by following the historical order it is possible to present the circumstances which led to the creation of a mathematical idea and to show the influence it exerted. Modern civilization and culture are an accumulation, an amalgamation, and a fusion of contributions from many earlier civilizations and cultures. The historical order of events permits us to break apart the whole complex of modern ideas insofar as they relate to our subject and to examine the contributions one by one.

I shall illustrate how some topics are treated. The material begins with some facts about mathematics in Egypt and Babylonia. The primary object of this topic is to present the origins of mathematics and its uses in relatively primitive civilizations. At the same time one can emphasize the empirical nature of pre-Greek mathematics in order to prepare the student for the change which takes place in the classical Greek period. In this chapter familiar facts about number are reviewed and accepted on the same basis as the ancient peoples themselves used, namely, experience.

We then proceed to the classical Greek period (600 B.C.–300 B.C.) and emphasize the change in the nature of mathematical activity introduced by the Greeks. Abstraction, deductive proof from explicitly stated axioms, and the emphasis on geometry as opposed to algebra are the salient changes and these can be related to the nature of classical Greek thought and society. In this same period I review a few facts about Euclidean geometry and then show that the outstanding characteristics of this creation are precisely those of Greek philosophy and Greek art. In addition, the significance of Euclidean geometry for later ages as a model of rigorous reasoning is pointed out.

The next topic is the mathematics of the Alexandrian Greek period, which lasted from about 300 B.C. to about 600 A.D. The mathematics of this period favored science and measurement. Trigonometry arose at this time as a step towards quantitative astronomy and so I teach the trigonometry of right triangles and the use of trigonometry in the measurement of the sizes and distances of the heavenly bodies. Other applications of the trigonometry of the right triangle such as navigation and the analysis of forces into components can be included.

The final topic in the Greek period centers about the Greek doctrines that nature is designed in accordance with mathematical laws and that man can fathom that design through the mastery of mathematics. The importance of Euclidean geometry and of the supreme Greek achievement in astronomy, Ptolemaic theory, as evidence for the rationality of nature, is discussed. This material also shows that mathematics made possible the first truly great astronomical theory and the first great scientific synthesis.

The transition from the Greek age to the Renaissance is made by giving a brief historical account of the state of affairs in medieval Europe, the role of mathematics in this period, and the importance for subsequent developments of the Catholic doctrine that nature is rationally designed by God and intelligible to man. It is also relevant to emphasize at this stage that a lack of interest in life in the physical world is detrimental to the growth of mathematics.

The historical movement known as the Renaissance is then sketched. The flow of Greek works to Europe gave the Europeans the opportunity to take up where the Greeks left off. The revived interest in the natural world suggested a host of problems. And the fusion of the Catholic doctrine that nature is the handiwork of God and therefore worthy of study with the Greek doctrine of the mathematical design of nature inspired the mathematicians and scientists to take up their chosen tasks with religious zeal.

Among developments of the Renaissance the first one treated is the creation and introduction of the heliocentric theory by Copernicus and Kepler. This particular creation involves no new mathematical ideas but it does give one a chance to show how, for purely mathematical reasons and despite weighty scientific and religious counter-arguments, the courses of astronomy, science, philosophy, and even religion were altered.

A second great development of the Renaissance originates with the work of the painters. These men sought to depict nature realistically and hence faced the problem of reproducing three-dimensional scenes on canvas. By applying Euclidean geometry to this problem the Renaissance artists developed the science of perspective. A few principles of this science are readily taught and the effect of this development on painting can be demonstrated by displaying and contrasting late medieval and Renaissance paintings. In the consideration of projection and section, the basic idea underlying their science of perspective, the painters raised questions which the mathematicians took over and answered by the creation now known as projective geometry. A few theorems of this geometry can be taught partly to enlarge the student's knowledge and partly to show that a vast and vital branch of mathematics was inspired by the art of painting.

The usual material of co-ordinate geometry is the next topic. However it appears to me to be very much worth while to lead into this subject by showing how Descartes, lost in the intellectual storms of the seventeenth century, turned to mathematics to find a new approach to truth. Descartes isolated the principles of mathematical method and applied his "universal mathematics" to philosophy. He then applied the method to the study of curves and came up with a much needed method of handling all curves. The potentialities in co-ordinate geometry for the study of new curves and higher dimensional geometry can now be discussed.

Like many other courses this one emphasizes the notion of functionality. The introduction of this concept is motivated by first discussing Galileo's new view of the role of science, namely, to describe natural phenomena quantitatively. Such descriptions are best made in terms of relationships among variables; hence the importance of the function concept. However instead of using artificial and disconnected examples of functions one can use the laws of motion and gravitation. Moreover, to show that functional relations expressed as formulas can be extremely helpful to scientists one can derive new laws of motion from Newton's laws and the law of gravitation. Derivations employing only elementary algebra and yet yielding significant conclusions can be given. It is then possible to show how successful these laws and their implications were in describing motions on the Earth and in the heavens. The flight of projectiles and the motions of the planets around the sun are encompassed. So far-reaching was the range of these laws that they were called universal laws. The role of the calculus in deriving universal laws, at least insofar as the concept of instantaneous rate of change is concerned, can be included in this general topic.

The remarkable evidence provided by the universal laws of motion for a mathematically designed and lawful universe was acclaimed by all European intellectuals and inspired a rationalistic movement. Imbued with the conviction that Reason, personified by mathematics, could solve all of man's problems, the great minds of the eighteenth century undertook a sweeping reorganization of science, philosophy, religion, ethics, literature, and the social sciences. It is possible to explain how science became more dependent upon mathematics, that doctrines such as materialism and determinism were built upon mathematical and scientific foundations, that the source and nature of truths were re-examined, that new religious movements culminating in Deism resulted, and that poetry was deprecated in favor of prose. All these topics can be presented in concrete terms. In a few cases they can be assigned as outside reading.

Returning to mathematics proper one could treat next the trigonometric functions and apply them to the mathematical analysis of musical sounds. Using sound waves as a concrete analogy one could also include a somewhat loose and largely qualitative account of electromagnetic phenomena. Though the mathematical theory of these phenomena proved to be invaluable for the design of the radio, the telephone, television, and other modern miracles of science, its value in organizing and interpreting a whole class of seemingly diverse natural phenomena is stressed. Further, since the physical nature of all electromagnetic phenomena and of radio waves in particular is completely unknown, it is possible to illustrate the meaning of the prevalent philosophical doctrine that in the last analysis our best scientific knowledge reduces to mathematical formulas. These are the nature of the physical world.

A major objective of the course is to introduce the fundamentals of statistics. However, it seems highly desirable to explain why statistical methods were sought and emphasized. Hence one might point out first that the rationalistic spirit of the Age of Reason had infused the social scientists with the desire to discover the universal laws of their domains. But the attempts to deduce these laws of man and society from *a priori* principles failed to produce realistic sciences. The social scientists then sought and created new mathematical techniques for the derivation of laws by working from statistical data. The way is now open to treat the elements of statistics and probability and to show how significant laws were derived with these techniques.

From the statistical techniques there resulted statistically likely conclusions which, nevertheless, seemed to apply as infallibly as did the mathematically deduced, necessary laws of the Newtonian era. A new philosophy, the statistical view of nature, arose to challenge the philosophy of determinism and has had grave implications for philosophy, religion, and science. These implications should be discussed.

We are now close to modern times and, in view of the rapid development of mathematics in this period, some selection of topics is forced upon us. Among many possibilities the notion of transfinite numbers and the value of this concept in providing a sound analysis of length, time, and motion seem to warrant inclusion. Even more important in this period is the remarkable creation of non-Euclidean geometry. Several modern texts now include this subject but why they avoid pointing out the implications of this creation is a mystery to me. In intellectual spheres no creation of modern times has been more revolutionary for, in effect, non-Euclidean geometry has taught us that there are no truths. Moreover, this creation caused a marked revision of our understanding of the nature of mathematics.

Without expecting to do any more than pander to popular interest one could show how non-Euclidean geometry is applied in the theory of relativity. The presentation should be, of course, purely qualitative and highly simplified.

The entire presentation closes with a discussion of the twentieth century's understanding of the nature of mathematics. The essential features, mathematics as a method of approach to knowledge and mathematics as an art, should be stressed.

It is perhaps needless to point out that many a good intention has gone awry in the execution. The handling of the above material must produce a coherent, rounded, sober presentation of elementary mathematics. There must be proper balance between the large ideas and concrete illustrations. There must be precise assertions and proofs. However, there is no need to prove all the conclusions. Moreover, the proofs must be distinguished from intuitively grounded and loose arguments. Finally, the treatment must be on a level suitable for freshmen.

6. Criticisms of the proposed course and rebuttal. The above course does seem to be a radical departure from conventional ones and in discussions with colleagues I have encountered criticisms which I should like to consider for a moment. One such criticism is that the course would amount to just a lot of vague talk, an ill-defined mass of material. One answer to this criticism could be that no course is vaguer as to the nature and role of mathematics than college algebra and trigonometry. However, more to the point is it that each topic has definite mathematical content and establishes definite relationships of mathematics to our culture. The critic who believes that the kind of course I have outlined would be vague evidently doesn't believe that mathematics has any ideas or significance and hence that there would be little to say about it. A course which emphasizes concepts and their influences can be made substantial by insisting upon a good understanding of what is taught.

Again, I have been told that some of the topics are too sophisticated to be grasped by freshmen. This criticism is justified in part. However, each course must open up new fields and new concepts which will be pursued and more fully understood through work in subsequent courses. That is, each course must break fresh ground to some extent. In later courses some of these ideas will be reconsidered and the student will progress farther in these courses because he already has some notion of what is being discussed. The mathematics course I have sketched would break ground for science and philosophy courses in particular. More than that, many colleges are now experimenting with broad humanities courses in history, literature, physical science, and the social sciences. Each of these approaches our culture through its own avenues. The mathematics course described above would fit excellently into such a program and reinforce the other courses.

Some teachers would argue that the course I have outlined is mathematically thin. Where, for example, is the quadratic formula? Of course these critics are begging the question. What is mathematics? Evidently to these people it is a series of techniques. It does not include the meaning, purpose, and significance of the technical material. These critics merely reflect their own failure to see mathematics broadly. Better one idea well understood as to meaning, purpose, and significance than a thousand techniques however well mastered. A virtuoso on the violin is not a musician.

Moreover, concepts are more difficult to grasp than mechanical procedures which call for only parrot-like responses. Any child could learn that if $y=x^2$ then $dy/dx=2x$. However, how many calculus students can describe the concept represented by dy/dx and how many can state what is accomplished by the calculus?

Then there are "practical" objections. Some teachers point out that a student can't go on from a course such as I have described to more advanced mathematics. My answer is this. If a student who has indicated at the outset that he does not intend to pursue mathematics or use it in later life changes his mind at the end of the course, then he should be willing to take a technical course such as college algebra and trigonometry before going on to the higher courses. In any case the 99% who won't go on should not be sacrificed to the one per cent who might.

I know that some chairmen object to innovations in freshman mathematics because it is supposed to be hard for the student to transfer credit from one college to another. Hence they permit innovations as long as the first semester contains college algebra and the second, trigonometry. Other chairmen object to my course because it cannot be taught by the graduate students who, in some places called universities, teach the bulk of the freshmen. Of course these chairmen are putting the cart before the horse or are just rationalizing their unwillingness to consider new ideas.

There is a very real practical difficulty and that is to find college teachers of mathematics who will undertake to present the cultural aspects of mathematics. For various reasons which I propose to discuss at greater length at another time, mathematics teachers pay least attention to their most critical teaching problem, namely, the education of the liberal arts student. However, the task of obtaining the proper teachers is merely the task of recognizing and re-orienting ourselves to the problem posed by the liberal arts student. It is not the task of finding persons with extraordinary talents.

The best answer to all dubious objections is experimentation. This we are now doing at New York University. The material I have sketched was tried in one section of about 30 students during the academic year 1951–1952. Two sections were taught this material during the year 1952–1953. Two more sections of this course are presently being used for this experiment. The general cultural material on which the course draws has been gathered and published under the title of *Mathematics in Western Culture* (Oxford University Press, N. Y., 1953).

It is too early to assess fully the results of the experiment. One positive accomplishment can be noted. The students understand what is being tackled in each topic and participate fully in the class work. They feel that this material is their meat, so to speak, and partake of it. Since the course makes contact with painting, philosophy, literature, the social sciences, and other fields, even the most disinterested student is drawn into participation at some time. The interest displayed contrasts sharply with the reticence, resistance, and helplessness one encounters in teaching the traditional material.

7. The major objective of the proposed course. What should be the prime accomplishment of the proposed course? Essentially it should teach an appreciation of the role of mathematics in Western culture. Appreciation rather than skill has long been recognized as an objective in literature, music, and art. It seems to me equally justifiable as an objective in mathematics, especially in view of the facts that interest in mathematics must be aroused and that the subject is more difficult to grasp.

If we are successful in teaching appreciation of mathematics we shall replace the present dislike and complete rejection of the subject on the part of students who have suffered through college algebra and trigonometry by respect and possibly liking for the subject. We may even succeed in inspiring them to maintain some contact with mathematics in later life and thereby secure for them one of the noblest of intellectual interests. We know that this interest will be well rewarded.

COUPON COLLECTING FOR UNEQUAL PROBABILITIES*

HERMANN VON SCHELLING, U. S. Submarine Base, New London, Connecticut

1. Introduction. Let us assume that a firm encloses in each package of a certain product one coupon which is chosen randomly from a stock of k different items. A customer who collects these coupons may gather some of them several times before he completes a whole set. The problem arises to determine the probability of obtaining all of the k coupons in n packages. An obvious generalization is what will be the probability that the lth different coupon is found in the nth package.

Since 1938 this question was treated by several authors [**1**], [**2**], [**3**] who always assumed that each coupon has the same probability $1/k$. The method was applied by M. G. Kendall and Babington Smith [**4**] for checking the randomness of their random sampling numbers.

In the general case the ith coupon has the probability p_i with $p_1+p_2+\cdots+p_k=1$. The solution of the general problem was published by the writer [**5**] in a German journal of a limited circulation as early as 1934. This paper remained practically unknown; the authors mentioned above did not refer to it. Besides, the formulas were given without proofs. Since the problem seems to be of a continuous interest, a short demonstration of the general case might be justified.

2. The general distribution. Let us assume k kinds of events which occur with the probabilities

$$p_1, p_2, \cdots, p_k; \qquad p_1+p_2+\cdots+p_k=1. \tag{1}$$

* Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.