MATHEMATICAL CONSULTANTS, COMPUTATIONAL MATHEMATICS AND MATHEMATICAL ENGINEERING

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1. Introduction. This note, and a companion (which may eventually appear on *The teaching of concrete mathematics*) are after-effects of the Conference on Training in Applied Mathematics held in New York on October 22–24, 1953.* They represent an attempt to combine, in unknown proportions, some implicit conclusions of the conference, as I sensed them, and some very personal feelings which have developed slowly over a longer period. The present note discusses attitudes which mathematicians and mathematics departments might take toward (1) the training of mathematical consultants, (2) the place of computational mathematics, (3) the place and role of groups working in theoretical mechanics, and (4) the emergence of Departments of Mathematical Engineering and of training in computation engineering.

Before going on, a word about the word "engineering." Some think it is a bad word—they like to be pure. Some think it is a good word—they like to be engineers. The writer associates with both pure mathematicians and engineers regularly and by choice—he thinks of it as a neutral word, and he hopes that his readers will do the same, at least while reading through and mulling over this discussion.

2. Mathematical consultants. Probably the most important single mathematical function in industry and government is that of the mathematical consultant. Today there are few mathematical consultants, tomorrow there will not be very many, but their importance will far outweigh their numbers.

On their training the writer has strong views, and, he believes, strong support for them. First, and foremost, it should be a research training, most commonly in pure mathematics, but not infrequently in physics, a mathematized branch of engineering, or even in applied mathematics. Secondly, it should involve as wide a background as possible.

The basic requirements for a mathematical consultant are threefold. In addition to the research training and the broad mathematical background, the mathematical consultant needs, as do consultants based on any scientific field, *above all else* an interest in the other man's problems—in these problems as wholes, not just in their mathematical aspects.

The future consultant will be trained along the same lines as the future research mathematician, and will be distinguished from him, not by a lack of research ability (for he needs as much as they do), but rather by his broader interests. His training asks only of Departments of Mathematics that:

(1) they encourage their students to browse around, and

* Bull. A.M.S., vol. 60, 1954, pp. 38–44.

(2) they consider it reasonable that some of their best students become mathematical consultants.

For mathematical *consultants* we don't want mathematical engineers, we *do want mathematicians* (or theoretical physicists turned mathematicians, or electrical engineers turned mathematicians, *etc.*)!

3. The place of computational mathematics. The writer makes a sharp and important distinction between the mathematics and the engineering of computation. The mathematics of computation is concerned with what happens as a result of various computational steps or procedures. The engineering of computation is concerned with how best to choose steps or procedure to obtain the desired results with relatively small expenditures of effort or money.

Thus, for example, computational mathematics recognizes *one* interpolating polynomial through a given set of points, and notes in passing that values of this polynomial can be found by Aitken's method, by forward or backward Newton's formula, by Everett's formula, and by many other specific expressions. Computation engineering is concerned with the advantages and disadvantages of the various methods, and with the choice between them in particular situations.

Computational mathematics is a branch of mathematics like any other, and is logically taught by mathematicians. Today many places teach a little at the undergraduate level, mostly mixed with considerable computation engineering. Tomorrow, a few institutions will have graduate programs preparing a few Ph.D.'s in computational mathematics. These Ph.D.'s will need to meet the requirements of mathematical ability and background currently required of Ph.D.'s in pure mathematics, and many of them will go on to do research in computational mathematics as teachers in colleges and universities.

The responsibilities of mathematics departments toward computational mathematics are, then, to be met by:

(1) readiness, nay even eagerness, to teach computational mathematics at the undergraduate level to all who wish to learn (in all departments), and

(2) provision of needed research support, which is needed by both computation engineering and the further development of computational mathematics, through the training of a few good Ph.D.'s in computational mathematics. (The need exists now, and will grow rapidly. The Ph.D.'s should fully meet the standards of pure mathematics. A few institutions could meet the present needs.)

4. Theoretical mechanics. To allow for the mental habits of many readers, we shall use the term "theoretical mechanics" for what would be better called, in the writer's opinion. "(classical) physical mathematics," where the order of the words (just as in "physical chemistry" or in "mathematical physics") implies that the subject is *mathematics* which is oriented toward physics. Besides all those things which are truly mechanics we include here electromagnetic theory and boundary value problems.

In view of the prevalence of the relation

applied mathematics $\equiv$ theoretical mechanics (modulo unimportant details)

in many minds, no discussion bearing on the applications of mathematics can avoid a discussion of theoretical mechanics sufficient for orientation. (In England the congruence of applied mathematics with theoretical mechanics seems to have been strengthened to an equality.) According to the recent NRC-AMS conference on applied mathematics in New York City, the recipe for theoretical mechanics in the United States seems to be:

> 2 parts mathematics, 1 part physics, 2 parts engineering, mix well and serve labelled "Applied Mathematics."

Why should this recipe be preferred?

The writer's estimate runs about as follows: At the present time, theoretical mechanics tends to be too routine for the science departments (physics and mathematics) and is almost surely too fundamental for the engineering departments (aeronautical, electrical, hydraulic, mechanical). Thus it has to be operated on an intermediate "applied science" level. In most institutions this is most easily done, if it is done at all, through a cooperative arrangement between science and engineering. This is quite likely to be a transient situation. If the engineering departments strengthen their fundamental side, as many of them indicate desires and plans to do, then theoretical mechanics may become a sophisticated branch of engineering. (After all, some engineers assert that, now the physicists have dived into the nucleus, they will have to take over not only theoretical mechanics, but experimental mechanics, optics, acoustics and the rest of classical physics. This seems unlikely in the long run.) Whether it will then be a branch of *mathematical* engineering or not is another matter, but if the ratio of mathematicians to physicists interested in theoretical mechanics continues to have its present high value, this will probably happen.

What, then, is the appropriate attitude of the mathematics department toward theoretical mechanics? As far as an undergraduate program is concerned, the answer is clear. It is best for the mathematics department to teach mathematics, for the physics department to teach physics and for the engineering departments to teach engineering. Unless there is an unusually strong base in the engineering department, a base so strong as to be extremely rare today, such a program is probably best guided by a joint committee. And the recipe: 2 parts mathematics, 1 part physics, 2–3 parts engineering, seems entirely natural for such a committee. This can be considered, without loss of generality for *internal* discussion among mathematicians, a program in mathematical engineering. (For discussion with other departments it will undoubtedly be necessary to emphasize the joint aspects.) As such it poses no new problems beyond those discussed in the next section.

At the graduate and postgraduate level, the situation is different. Here there must be strong research interests in theoretical mechanics. These research interests are not (with rare exceptions) housed in engineering, in physics or in mathematics. There must be a house for them somewhere. Hence the development of graduate institutes in the area (*e.g.*, Brown, Indiana, Maryland). Toward such a group what should be the attitude of the mathematics department? Clearly those groups function, at those instants when they function mathematically, at a more fundamental level than, at present at least, any of the conventional engineering departments. Thus they are closer relatives. On the other hand, they are not in the immediate family. The appropriate relation would seem to be that of double first cousins—very close, but not quite inside the family. (Hence one must be doubly careful of the Persian proverb "To hate like cousins!")

The most obvious outward implications of such an attitude would seem to be these:

(1) encouragement of joint appointments whenever the appointee would fit into the mathematics department;
(2) acquiescence, for the present, in a title for such a group involving the words "applied mathematics" if that is what the group wishes,
(3) continuous, but gentle, education about the advantages to both parties of a title which expresses the character of the group more clearly and accurately.

In brief: support and cooperation but avoidance of blending.

5. Science, mathematics and engineering. Every mathematician acquainted with the history of United States colleges and universities which have had two mathematics departments, whether the second be called a department of applied mathematics or a department of engineering mathematics, fears such arrangements. For in almost every case, they have led to watering down the quality of mathematics and to ill feeling. There is a natural tendency to carry over this feeling to any administrative arrangement where all the "mathematics" is not in one mathematics department. This natural tendency may not always be wise, as we shall try to show.

The writer believes, indeed, that chemistry departments have gained rather than lost from the presence of chemical engineering departments and that physics departments have gained rather than lost from the existence of mechanical, electrical, and aeronautical engineering departments. He believes, furthermore, that most members of chemistry and physics departments would agree to this. Let us try to inquire into why this is so.

There was then, and surely continues today, a serious need for chemical, mechanical, electrical and aeronautical engineers. Three ways of meeting this need are conceivable (we shall state them in terms of physics alone for convenience):

(1) The physics department could have taught both physics and engineering.
(2) The physics department could have taught physics and the engineers could have taught engineering, or

(3) The engineers could have taught both physics and engineering.

Of these, (2) was generally selected. The writer submits that this was the best choice for the physics department. Why?

If (1) had been selected, what would have happened to physics departments? They would have faced the teaching of courses in alternating current machinery, bridge design and propeller theory, to name some examples. Such courses would have been boring to the permanent staff, but would have required too much background to be safely turned over to fresh Ph.D.'s. Either the morale and research potential of the permanent staff or the reputation of the training would have gone down, down, down.

If (3) had seen selected, what would have happened to physics departments? At first glance, the main penalty would have been a size and support penalty due to a loss of students. But actually the situation would have been far worse in the long run. The wholly engineer-taught engineer would in the long run have lost out on fundamental physics. And in the longer run, this would have been evident in comparison with the increasing demands placed on the engineer. As a consequence, the physics departments would have to shoulder the responsibility of providing more fundamentally trained engineers and the difficulties associated with (1) would not have been avoided.

Either (1) or (3) would have a fate worse than (2) *for physics departments*—and, incidentally, either would, of course, have provided engineers with poorer training. Perhaps this example has a moral for mathematics departments!

This moral, if it exists, must be this: "When the demand for mathematical engineers begins, plan to meet it cooperatively—plan to teach the relevant mathematics in the mathematics department—plan to teach the mathematical engineering elsewhere." The writer believes that this moral exists and is important. He hopes that mathematics departments will preserve both their research potential and their student-semesters of classes by adopting this policy when it becomes timely.

6. What is mathematical engineering? Many readers may by now be muttering under their breaths, "But what *is* mathematical engineering?". To this question we have no complete answer, any more than Faraday could explain the uses of newly discovered electromagnetism. But we can and will give partial answers.

Mathematical engineering consists of those branches of engineering where the single most important tool is mathematics. This is the natural definition. Does it help us? Certainly it helps us a little, for it makes it clear that the engineering of computation is certainly a branch of mathematical engineering, for surely in planning computation—in planning mathematics—mathematics is the one most essential tool. In those parts of theoretical mechanics too, where the physics and the differential equations are long known and firmly settled, the most important single tool may well be mathematics, and, where it is, these parts of theoretical mechanics may prove to be mathematical engineering. (In the area now labelled industrial engineering, some further branches are probably being shaped, but it would be premature to try to identify them here.)

First to be of importance is undoubtedly that which is first above—computation engineering. It is here that the first great needs will come.

7. Computation engineering. Today the need for mathematical engineers has begun to appear. The first need is for computation engineers! The arrival of the first batch of IBM type 701 high-speed calculators (formerly known as Defense Calculators, latterly as Electronic Data Processing Machines) is a substantial indication. The estimate of a staff of 30 persons per machine for economic balance is almost certainly conservative. But this means that over 500 trained persons are required for these machines alone.* What kind of persons?

Today the man with a 701 seeks out A.B.'s or M.A.'s in mathematics for his coders and lower level problem analysts, who will form 90% or more of his staff. The work he has for them is high-level routine and requires a solid grasp of a substantial amount of mathematics. Qualitatively, it seems to the writer entirely comparable to the work of an average mechanical or electrical engineer. As time goes on, are not the increasing numbers of such jobs going to be filled by mathematical engineers, whatever they may be called? How can it be otherwise?

Today the demand for computation engineers is definitely here. Tomorrow it will be larger. Today a few institutions will set up sources of supply. (How many? This is not clear.) Tomorrow, more and more will join them. (How fast? This is far from clear.) What should mathematics departments do, or be prepared to do?

If the moral drawn above is correct, then mathematics departments should, in the writer's judgment, act as follows:

(1) They should prepare to cooperate in the setting up of training for computation engineers under engineering auspices, and

(2) They should prepare to teach the necessary mathematics, *including the mathematics of computation*, in the mathematics department.

8. Summary. In the writer's opinion, then, Departments of Mathematics should:

(1) encourage at least some of their students to "graze around"

(2) look forward to a very few of their best students becoming mathematical consultants in industry or government

(3) stand ready to teach the mathematics of computation to all who wish to learn

(4) in a few cases, start training high-caliber Ph.D.'s in computational mathematics

(5) encourage joint appointments with theoretical mechanics groups whenever this is reasonable

(6) acquiesce, for the present, in a title of "applied mathematics" for such a group

* *Note added in proof.* In the year or year and a half since these words were first written, the predictable demand has increased by a *factor* of at least five, or more nearly ten!

(7) point out the advantages to both parties, gently but steadily, of a more appropriate title
(8) prepare to cooperate in the setting up of training in computation engineering under engineering auspices, and
(9) prepare to teach the necessary mathematics in the mathematics department.

Through such policies can Departments of Mathematics maintain and increase their strength and quality.

MATHEMATICAL NOTES

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NOTES ON MATRIX THEORY—VI

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1. Introduction. In two recent notes, [**1**], [**2**], we have shown how various results relating to the determinant of A, a positive definitive matrix, could be derived from the well-known identity

$$\frac{c_n}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} dV_n \tag{1}$$

where the integration is over all x, and $c_n = \sqrt{\pi}^n$, a constant depending only upon the dimension n.

If we define, for $k=1, 2, \cdots, n$,

$$|A|_k = \prod_{i=1}^{k} \lambda_i \tag{2}$$

where λ_i, $i=1, 2, \cdots, n$, are the characteristic roots of A, a positive definite matrix, arranged in increasing order of magnitude, it was shown by Ky Fan, [**3**], [**4**], that

$$|A\lambda + B\mu|_k \geqq |A|_k^{\lambda} |B|_k^{\mu}, \qquad \lambda, \mu \geqq 0, \lambda + \mu = 1. \tag{3}$$

This result, together with some additional results, was recently obtained in a different fashion by Oppenheim, [**5**].

The purpose of the present note is to establish an identity for $|A|_k$ similar to (1). This identity may then be utilized to derive (3) in the same manner that