THE TEACHING OF CONCRETE MATHEMATICS

JOHN W. TUKEY, Princeton University and Bell Telephone Laboratories, Inc.

1. Introduction. One syndrome must, from time to time, disturb the sleep of all concerned with the applications of mathematics,—a syndrome never discussed in open meeting, perhaps because of its sensitivity. It seems to be generally agreed that "applied mathematics" is more difficult than "pure mathematics" in requiring more maturity and more years of study before useful results are attained. Today's leaders in "applied mathematics" were mainly trained in "pure mathematics." Yet from the point of view of *research potential and related intellectual ability* the students who *study* in "applied" fields do not compare in strength with those who go into "pure" mathematics! Is this not a paradoxical situation?

One can try to make the situation appear less paradoxical by going further, and asserting that: "Just as today's leaders in the applied fields have come mainly by conversion from the pure, so too will tomorrow's!" (A statement which is undoubtedly true for tomorrow!) But what of the day after tomorrow? Should conversion be inevitable? If it is not, as the writer believes, then the answer must lie in the early training of our students.

Two causes deflect students from the "applied" to the "pure" today:

(1) a feeling among teachers that the "applied" is beneath the "pure,"
(2) a failure to present the "applied" so that it is as intellectually stimulating as the "pure."

Given the temperaments and intellectual orientations of collegiate teachers of mathematics, it is clear that (1) is an inevitable consequence of (2) and that direct (or slanting) attacks on (1) are useless. To improve the situation we must deal with (2), when (1) will, more or less slowly, take care of itself.

How then, may we make "applied" or "concrete" mathematics more stimulating? Many ideas may be needed in the long run, but here are some which appeal strongly to the writer:

(a) we may strive to develop the areas of formulation and approximation, where applied mathematics has failed to heed the admonition "physician, heal thyself."

Success here could give us something of a truly mathematical nature worthy of being taught as applied mathematics. At the best, this is a long-range program (and some may term it visionary)—it is discussed briefly in Section 7.

(b) we may introduce more generality into each stage of the teaching of concrete mathematics—for example, after meeting one or two expansions into eigenfunctions, we may give a nonrigorous introduction to eigenfunction expansions in general.

This program could begin tomorrow, or even today, and needs no detailed spelling out here. It will only succeed, however, if it is focussed on concepts rather than rigor. Usually the physicist or other user will not only omit all rigor in his justified haste to treat a particular point, but he will omit or gloss over many concepts of general interest and help. It is for the mathematician to introduce the student to the majority of the concepts first, letting the rigor wait till its appropriate time. New concepts should be injected into the student as gently as possible!

(c) we may emphasize the study of computational procedures in their own right by discussing their general properties—the mathematics of computation—rather than merely grinding through them.

All of these changes require teaching time and student's time. If we are realistic, we must find time, or at any rate most of it, by saving it elsewhere. Where is time now occupied? With the mechanics of computation, numerical and algebraic. How can it be freed? By reorienting our attitude toward computation—by trying to make it less of a road block.

The key to the immediate attack, then, lies in our attitude and practices concerning computation, taken in a most general sense, and its techniques.

As we succeed with such a program, shifting emphasis from avoidable labor of computation to broader concepts on the one hand and the mathematics of computation on the other, we shall be teaching better 20th-century mathematics—better for mathematicians to teach—and better for students to learn.

2. Attitudes toward computation. Computation may be numerical or "algebraic" where the latter term seems in practice to cover all forms of systematic manipulation which are not merely numerical—polynomials, trigonometric functions, indefinite integration, summation (not summability) of series, tensors and logic, to name a few, all have algebras in the sense of orderly procedures of computation. In their essentials, the practices of these diverse forms of computation are the same. Given input data, one performs more-or-less-or-much-less routine operations with the intent of reaching output results of a predetermined form. Interest centers in the certainty, efficiency, and ease of manipulation of the operations. A certain amount of practice is useful, both to promote understanding (which is not helped appreciably by still more extended practice) and to provide a little facility of manipulation (usually a little suffices).

Numerical computation, through the centuries, has often faced up to reality and made things easier. The use of logarithmic tables, even by those who do not know how to recompute them, and of desk calculators and, now, electronic calculators, even by those who cannot repair them, has been a commonplace. Today the "software" comprising the carefully planned interpretive routines, compilers, and other aspects of automative programming are at least as important to the modern electronic calculator as its "hardware" of tubes, transistors, wires, tapes and the like. When a student or a user begins to use an electronic calculator, we do not ask him to learn all the details of the automatic programming—and surely not to learn why these details were chosen instead of others. A few students and users will develop slowly into designers or programmers, but their number will be few and their treatment special. Let us look to the analogy in all forms of computation.

Throughout computation, since the usual student will not compute steadily, but rather occasionally, even if he continues to compute all his life, emphasis is best laid on methods which are easy to remember—and, more importantly still, easy to relearn when forgotten. We are really concerned with the continuing capabilities of the student, including those which require supplementation by some re-study when used, rather than with student behavior in a final examination.

There are some students (but how few!) who will go on to compute steadily. They require special training, but their needs should not prejudice the training of the larger student body. We do the specialists no injustice to teach them the easy-to-remember way first, even though it may take 10%, or 20%, or 50%, or even 100% longer than the fanciest method when in steady use. This will not dull their interest in, and appreciation of, the fast, hard-to-remember methods. But if we teach the hard-to-remember method first, the occasional computer will never get to the easy-to-remember method, never have a method he can use when he meets a real problem, and thus never solve the problem.

3. An example from numerical computation. But, some may say, teaching easy-to-remember methods means teaching technique, and ideas and concepts will suffer. This is not so. Let us take Aitken's method of interpolation, [**3**], [**4**], as an example. The basic problem is to pass a polynomial through given points (x_1, y_1), (x_2, y_2), (x_3, y_3), and so on. If $P(x)$ with numerical subscripts represents a polynomial passing through the points indicated by the subscripts (thus $P_{124}(x)$ passes through points 1, 2 and 4), and if "$-$" stands for any collection of subscripts other than i or j then

$$[(x - x_i)P_{-j}(x) + (x_j - x)P_{-i}(x)]/(x_j - x_i)$$

passes through all the points corresponding to "$-$" and through points i and j. The argument which shows this is simple, direct, and truly mathematical. Hence we may define

$$(x_j - x_i)P_{-ij}(x) = (x - x_i)P_{-j}(x) + (x_j - x)P_{-i}(x)$$

and starting with

$$\begin{aligned} P_g(x) &= y_g, \\ P_{gh}(x) &= [(x - x_g)y_h + (x_h - x)y_g]/(x_h - x_g) \\ &= \frac{y_h - y_g}{x_h - x_g}\,x + \frac{x_h y_g - x_g y_h}{x_h - x_g}, \end{aligned}$$

obtain all the interpolating polynomials we may desire.

If all we wish is the value of the interpolating polynomial at a given x, then the process reduces itself to successive linear interpolations; thus, for $x=6.1$ and a particular array of x_i and y_i, we have:

i	x_i	x_i-x	$y_i=p_i$	p_{1i}	p_{12i}	p_{123i}	p_{1234i}
1	7	0.9	.84510				
2	5	−1.1	.69897	.77935			
3	8	1.9	.90309	.79391	.78469		
4	4	−2.1	.60206	.77219	.78723	.78580	
5	10	3.9	1.0000	.79863	.78359	.78573	.78578

Every computation here is straightforward linear interpolation or extrapolation. Thus

$$.77935 = \frac{(0.9)(.69897) + (1.1)(.84510)}{2.0},$$

$$.78723 = \frac{(-1.1)(.77219) + (2.1)(.77935)}{1.0},$$

and so on. Undoubtedly this is the easiest of all polynomial interpolation schemes to learn or relearn. (Even though, in the hands of the professional computer, it may be a little slow by comparison with some other schemes, its nearly-iterative and checking features are quite valuable.)

It is easy to learn, yet its teaching is not mainly teaching technique. What has to be taught, in order that its functioning be understood is not technique, but rather (i) that there will be an interpolating polynomial, (ii) that linear interpolation between two equal values returns the same value, (iii) that linear interpolation at an endpoint returns the given value. The algorithm for the interpolating polynomial now follows, and from it the numerical algorithm. After learning a few things of mathematical content (he may even be led to try other operations in place of linear interpolation) the student is equipped to do polynomial interpolation of any order, direct or inverse (for no properties of the spacing of the x_i were used), without the need to recall or look up any coefficients.

The more computation that we can teach in such a form, the better—both for applied mathematics and for pure mathematics.

4. An example from algebraic computation. Formal integration is another example of computation. It points a road we should travel, quite a different road from that just indicated. Some teachers of calculus seem to fear integral tables, apparently feeling that their students should not only be able to develop all the elementary formulas, but should have had to do each several times! What is this but teaching unnecessary technique? In Newton's day these formulas were new and interested mathematicians. Today they are of use, rather than interest. So why should we not strive to make them useful? This means learning how to use integral tables, rather than how to derive them.

Some fear the development of "handbook engineers"—persons who cannot operate without handbooks. This fear is complex; *i.e.*, it has both real and imaginary parts. The ex-student who cannot integrate $\int x^7dx$ or $\int\sqrt{1-x^2}xdx$ with relative ease would be a discredit to the mathematics department, but need inability to find $\int(1+x^2)^{-3/2}dx$ or $\int\sin^2\theta\cos^4\theta d\theta$, without either a handbook or considerable pain, matter? The writer cannot see that it does. (He himself can work out the transformations, but would rather walk across the hall and borrow an integral table. Does this make him less of a mathematician—or only less of a computer?)

If the time that would otherwise be spent in learning how to derive integration formulas were diverted, not away from "mathematics," but to the introduction of additional mathematical ideas, we should make a great gain. In part, this could be done within the integral tables themselves. Consider the treatment of $\int x^nf(x)dx$ for some relatively simple $f(x)$ for which all such integrals are elementary. The most classical integral tables gave formulas for $n=1$, $n=2$ and usually a reduction formula for lowering n by one or two units. Then came tables which gave the explicit result for $n=1$, $n=2$, $n=3$, and $n=4$ before the user had to resort to the reduction formula. Next perhaps to $n=6$, and so on. The more extended tables are more useful, but some mathematicians find them overelaborate.

Why has no one taken the logical next steps? First, the value of $\int x^nf(x)dx$ will be a finite series. We do have notation, including the use of summation signs, with which to represent finite series. Can we not give the worked-out form for $\int x^nf(x)dx$ for general n in nearly every case?

Would this not be much more useful than a reduction formula? Why should user after user have to go through the same operations to deduce the same finite series from the reduction formula? Not only might we save labor by giving the finite series once and for all, but it is quite likely that we might hint successfully, to students and to users, that generality can mean less work. As mathematicians we should favor such hints.

Second, there is a deeper opportunity. Our integral tables give $\int x^nf(x)dx$, and in some applications we may get such expressions, but in others the user is concerned with $\int P(x)f(x)dx$ where $P(x)$ is a particular polynomial. Once we realize that $\int x^nf(x)dx$ corresponds to a finite series, it follows that $\int P(x)f(x)dx$ is also a finite (double) series. If we invert the order of summation, we shall usually find the answer to be a finite series, each of whose coefficients is of the form $b_{j0}a_0+\cdots+b_{jm}a_m$, where $P(x)=a_0+a_1x+\cdots+a_mx^m$. In more abstract terms, the vector of final coefficients is obtained from the vector u_i of polynomial coefficients by multiplication with a constant matrix b_{ji} with numerical entries. We need not use those fearsome words—but we can tabulate the numerical values of the b_{ji} and explain how to use them. To the extent that students and users make use of tables of b_{ji}, they are being introduced to the practice of matrix computation, and, implicitly, to the idea of a linear transformation.

We could use an integral table, were it rightly constructed, to introduce its users to such important mathematical ideas and notations as matrices, summation sign technique, and linear transformations. If these things come in as an alternative method—one not taught in class, but acceptable in home work or examination—as an alternative method which *saves work*, they will have by far the greatest chance of penetrating the indifference of the student or user not yet awakened to mathematics.

If a good table of indefinite integrals on this pattern takes 500 handbook-sized pages—with a thumb-index and keys like a book on birds or fishes—should we complain? Or even feel badly? The writer has a waiting, gaping vacancy on his desk for such a volume, which might appropriately be called "Integral Tables for the Occasional Integrator"—as do many of his colleagues. The student need not have to have all 500 pages—we can make up a student's 50-page version, containing the first 30 pages, one page in 7 for the next 70, and one page in 40 thereafter (it would not harm the student or weaken the effect if its paging showed the gaps). But the user would have a place for all 500 pages. If the table were laid out and explained properly, a student who had once learned to use it could relearn readily, as an ex-student, just what he needed in a specific situation. Such a person would not be looked down upon as a "handbook engineer," but rather looked up to as a user of mathematics who was controlling his computational problems, and not letting them control him.

5. Some nonexistent examples. There are other sorts of computation—and most of them lack even the beginnings of the tables and handbooks which would make their use easier, or even easy. A complete catalog would be lengthy, but some examples may be illuminating.

The National Bureau of Standards, with support from the National Science Foundation, has undertaken the preparation of a table of functions. Some think of this as a revision of Jahnke and Emde's table [1], now nearly 50 years old, but others think of it as more nearly the first approximation to a "Numerical Tables for the Occasional Figurer." It may well show a substantial amount of this last aspect, and, to the extent it does, it will tend to make numerical computation less of a bar to the applications of mathematics.

Consider ordinary differential equations. What is "ordinary" about them from the user's standpoint? The writer knows of but one table of solutions [2], and has no reason to be tremendously encouraged about its usefulness. Yet there are a number of possibilities. Our books on intermediate differential equations discuss the reduction of second-order linear equations to standard form—yet, though a number of second-order linear equations have been solved, who has ever seen a table of solutions for such equations after reduction to standard form. Why should there not be such a table?

Our books on elementary differential equations have followed for 60 years the mold of Murray's book ([5], 1897)—a book written but two years after Cantor introduced the union of two sets as a formal operation. Such books contain some relatively widely useful methods of solution, and some special methods which amused 19th-century mathematicians. Today they tend to say a little more about solutions in series, and, even, numerical solutions. If we had a "Differential Equation Table for the Occasional Integrator," we could condemn the minor techniques to the lumber room, and instead teach vastly more important, suggestive, and stimulating topics such as solution in series and numerical solution.

Modern mathematics devotes a great part of its attention to linear expansions of one sort or another. And many linear expansions are of great practical importance. The process of finding coefficients can almost always be regarded as a process of biorthogonal expansion—yet who has seen even a little table of biorthogonal expansions, to say nothing of the "Biorthogonal Expansion Tables for the Occasional Expander."

Let the reader continue the list.

6. Sources. Where are the better integral tables, the usable differential equation tables, the first biorthogonal expansion tables, and all the others, to come from? The writer does not know, but—there are many members of the Mathematical Association of America who are competent mathematicians, fitted to do original work, yet not stimulated enough by current abstract mathematics to be carrying on important research. If only a small portion of them were to consider the interest and reward associated with trying to make these forms of computation simple, general, *and* easy to learn or relearn, there would be hands enough to do much. (Many of these tasks could well be done cooperatively by substantial groups.)

The successful completion of such tasks would do much to aid the healthy and mutually supporting growth of pure and applied mathematics in America—let us hope that they will be completed.

7. Formulation and approximation. Finally, a word about two areas where we have not explored far enough to see which way we should follow—but which we should clearly attack and exploit—formulation and approximation.

It is agreed that formulation of the problem is usually the most important stage in "applied mathematics," just as insight into what theorem is true and (probably) provable is often the most important stage in "pure mathematics." In each case the formation of new concepts or the refinements of old concepts is likely to be an essential step. Insofar as a concept-former is a philosopher, all mathematicians need to be philosophers (of a very special sort).

The formulation of the problem is of the essence—yet who has studied the problem of formulation, who has tried to explain it to the student? Pólya wrote "How to Solve It"; who will now write "How to Formulate It"? Probably no one person can do it; many must work together, almost all of whom must be mathematicians—though they will usually have other skills as well. Studying the problem of formulation, formulating better and better approximations to it, finding useful concepts for its treatment—these are tasks for "applied mathematicians" skilled in formulation.

It is easy to argue that a book on "How to Formulate It" will be empty of real intellectual content. It would have been easy to argue 30 years ago that a book on the theory of games would lack intellectual content. In the past 30 years many relevant and useful concepts *have* been formulated in game theory. In the next 30 years many relevant and useful concepts *could* be formulated in formulation theory. Perhaps it is time to start, even if the task may be harder.

If we tried to write "How to Formulate It" today we would strike mainly questions: Where can we find examples to adequately set forth the principles? (What are the principles, anyway?). What kind of exercises can be used as homework? And so on, and on, and on. Yet, if "applied mathematics" is to grow properly, if there is to be something teachable and worthy of the name "applied mathematics," someone must tackle this problem—and eventually there must be developed a technique of wide usefulness and acceptability for teaching. This will not be easy, but it is badly needed.

What of approximation? Why is it paired with formulation? It, too, is a major stage in "applied mathematics," a matter of tactics rather than strategy perhaps, but surely a major stage. Without good approximations we should be lost, yet who knows what concepts are important in approximation? (From a routine mathematics course one would feel that taking the first terms of some series, power or Fourier or perhaps something more complex, was the natural approximation when only a few terms were permissible, yet this is often very wrong! How many connect the $(C, 1)$ summability of most Fourier series with practical approximations?) Yet a reasonable number of concepts have already been isolated, and are to be found by looking in corners. Many more concepts are undoubtedly near the surface. A discussion of approximation from the point of view of concepts and principles rather than labor would undoubtedly bring out much that was worthwhile. Here is another area for the formulators!

Who will begin to study approximation theory, rather than the theory of various specific approximations? Who will try to collect the important concepts in approximation, and try to add to them? What will be the results?

The suggestion of this section is merely this: Just as there is an applied mathematics of games, genetics, and mechanics, so there should be an applied mathematics (at least in terms of concepts, perhaps with techniques and operations) of the applications of mathematics. When there is, mathematicians will be able to teach "the applications of mathematics." At present only individual applications can be taught (and it is not likely to be too good for pure mathematicians to teach applications).

8. Summary. In brief we have said and argued that:

(1) the teaching of any form of computation should be directed (in relatively elementary or general courses) toward the occasional computer rather than the steady computer;

(2) this requires emphasis on simple, easy-to-learn-or-relearn methods;

(3) such emphasis need not mean emphasis on techniques, but rather may, and should, bring in more mathematical ideas than its converse, as exemplified by Aitken's interpolation procedure;

(4) the use of integral tables in teaching elementary calculus should be greatly broadened;

(5) integral tables could be redesigned to bring in the beginnings of matrices, finite summation, *etc.*, as methods of saving labor;

(6) many other forms of table, *e.g.*, solutions of differential equations in standard form, biorthogonal expansions, are badly needed;

(7) there are mathematicians competent to develop such tables who the writer feels would gain enjoyment and satisfaction from their preparation;

(8) a reduction in the labor of computation is the only visible way of finding the time and effort to make the study of computation more rewarding;

(9) there is much to be done in connection with the development of applied mathematics of formulation and approximation.

References

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3. W. E. Milne, Numerical Calculus, Princeton, 1949.

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ON THE LIMITING EQUILIBRIUM OF n MASSES

D. J. BENNEY, Canterbury University College, New Zealand

1. The general problem. A rigid framework rests on a rough horizontal plane (coefficient of friction μ), being in contact with the plane at the $n+1$ points $A_0, \cdots, A_n$. Let the normal reactions at these points be $W_0, \cdots, W_n$, respectively. A horizontal force P is applied at A_0 making an angle θ with a fixed direction, the magnitude of P being gradually increased until equilibrium is about to be disturbed. The initial displacement of the framework can be represented as a rotation about an instantaneous center I. Our problem is to investigate possible positions for I, and, in particular, to consider whether I may coincide with any of the points A_r. If I coincides with A_r then friction will be limiting at each point A_s, $s \neq r$, while if I is distinct from A_r the friction will be limiting at all points A_s.